Another Jump Inversion Theorem for Structures

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Abstract. In this paper we investigate the question of existence of a jump inversion structure for a given structure $\mathcal{A}$ in the context of their respective degree spectra and the sets definable in them by computable infinitary formulae. More specifically, for a countable structure $\mathcal{A}$ and a computable successor ordinal α, we show that we can apply the construction from [4] to build a structure $\mathcal{N}_\alpha$ such that the sets definable in $\mathcal{A}$ by $\Sigma_1^{c,\Delta^0_\alpha}$ formulae are exactly the sets definable in $\mathcal{N}_\alpha$ by Σ^c_α formulae.

1 Introduction

We shall work with abstract structures of the form $\mathcal{A} = (A; R_0, \ldots, R_{s-1})$, where A is countable and infinite, $R_i \subseteq A^{n_i}$. We use the letters $\mathcal{A}$, $\mathcal{B}$ to denote structures and the letters A, B to denote their respective universes.

We call f an *enumeration* of the set A if f is a partial one-to-one mapping of $\mathbb{N}$ onto A. We say that f is an enumeration of the structure $\mathcal{A}$ if f is an enumeration of its universe A.

If f is an enumeration of A and $R \subseteq A^n$, we denote $f^{-1}(R) = \{\langle x_1, \ldots, x_n\rangle \mid x_1, \ldots, x_n \in Dom(f) \;\&\; (f(x_1), \ldots, f(x_n)) \in R\}$. For $\mathcal{A} = (A; R_0, \ldots, R_{s-1})$ we define the total function $f^{-1}(\mathcal{A})$ in the following way:

- if $u = \langle k, v\rangle$ and $k < s$, then $f^{-1}(\mathcal{A})(u) = i$ iff $f^{-1}(R_k)(v) = i$, for $i \in \{0, 1\}$;
- if $u = \langle k, v\rangle$ and $k \geq s$, then $f^{-1}(\mathcal{A})(u) = 0$.

We call $f^{-1}(\mathcal{A})$ a copy of $\mathcal{A}$.

Richter [5] initiates the study of the notion of the degree spectrum of a countable structure.

Definition 1 *The degree spectrum of the structure $\mathcal{A}$ is the set of Turing degrees*

$$DS(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \text{ computes a copy of } \mathcal{A}\}.$$

For a computable ordinal α, we define the α-th jump degree spectrum of $\mathcal{A}$ to be

$$DS_\alpha(\mathcal{A}) = \{\mathbf{a}^{(\alpha)} \mid \mathbf{a} \in DS(\mathcal{A})\}.$$

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The notion of degree spectra gives us one way to compare structures. That is, for structures $\mathcal{A}$ and $\mathcal{B}$ and computable ordinals α, β, we ask whether $DS_\alpha(\mathcal{A}) = DS_\beta(\mathcal{B})$.

Now we give an informal definition of the set of *infinitary* Σ_α formulae in the language of $\mathcal{A}$, denoted by Σ_α. The Σ_0 and Π_0 formulae are the finitary quantifier free formulae. For a computable ordinal $\alpha > 0$, a Σ_α formula $\varphi(\overline{x})$ is an infinitary disjunction of a set of formulae of the form $\exists \overline{y}\psi$, where ψ is a Π_β formula, for $\beta < \alpha$, and $\overline{y}$ includes the variables of ψ which are not in $\overline{x}$. The Π_α formulae are the negations of the Σ_α formulae. The *computably infinitary* Σ_α formulae, denoted Σ^c_α, are those Σ_α formulae whose infinitary disjunctions are over c.e. sets. By $\Sigma^{c,X}_\alpha$ we mean the computable relative to the set X infinitary formulae. We refer the reader to [1, chap. 7] for more background information.

A set $X \subseteq A$ is Σ^c_α *definable* in the structure $\mathcal{A}$ if there is a Σ^c_α formula $\psi(x, \overline{y})$ and a finite number of parameters $\overline{a}$ in A such that $b \in X \leftrightarrow \mathcal{A} \models \psi(b, \overline{a})$. We denote by $\Sigma^c_\alpha(\mathcal{A})$ the family of all sets Σ^c_α definable in $\mathcal{A}$.

The notion of definability gives us another way to compare structures. That is, for structures $\mathcal{A}$, $\mathcal{B}$ such that $A \subseteq B$ and computable ordinals α, β, we ask whether $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathcal{A}) \leftrightarrow X \in \Sigma^c_\beta(\mathcal{B})]$.

For simplicity, in most of the constructions that follow we shall consider only structures of the form $\mathcal{A} = (A; R)$. In the end it should be clear that these constructions can be generalised to structures in any finite or effectively listed relational language. The next definition gives us the scheme that we follow to define our jump inversion structures.

Definition 2 ([4]) *Given a structure $\mathcal{A} = (A; R)$, $R \subseteq A^n$, and a pair of structures $\mathcal{B}_0$, $\mathcal{B}_1$ for the same relational language, let $\mathcal{N} = (A \cup U; A, U, Q, \dots)$, where*

1) $A \cap U = \emptyset$;
2) Q is an $(n+1)$-ary relation which assigns to each n-tuple $\bar{a}$ in A an infinite set $U_{\bar{a}}$, where $x \in U_{\bar{a}}$ iff $\mathcal{N} \models Q(\bar{a}, x)$. We also want $\bar{a} \neq \bar{b} \leftrightarrow U_{\bar{a}} \cap U_{\bar{b}} = \emptyset$;
3) The sets $U_{\bar{a}}$ form a partition of U;
4) Each of the other relations of $\mathcal{N}$ (in $\dots$) corresponds to some symbol in the language of $\mathcal{B}_0$, $\mathcal{B}_1$, and is the union of its restrictions to the sets $U_{\bar{a}}$;
5) For each n-tuple $\bar{a}$ in A, if $\mathcal{U}_{\bar{a}} = (U_{\bar{a}}, \dots)$, then

$$\mathcal{U}_{\bar{a}} \cong \begin{cases} \mathcal{B}_1, & \text{if } \mathcal{A} \models R(\bar{a}) \\ \mathcal{B}_0, & \text{if } \mathcal{A} \models \neg R(\bar{a}) \end{cases}$$

For a set of natural numbers X and a computable ordinal α, we denote by $X^{(\alpha)}$ the α-th Turing jump of X. Moreover, we define

$$\begin{aligned} \Delta^0_{\alpha+1}(X) &= X^{(\alpha)}, \text{ if } \alpha < \omega, \\ \Delta^0_{\alpha+1}(X) &= X^{(\alpha+1)}, \text{ if } \alpha \geq \omega, \\ \Delta^0_\alpha(X) &= \bigcup_p \{\langle y, p\rangle \mid y \in \Delta^0_{\alpha(p)}(X)\}, \text{ if } \alpha = \lim \alpha(p). \end{aligned}$$

We write Δ^0_a for $\Delta^0_a(\emptyset)$.

Although not explicitly stated as a theorem by Goncharov, Harizanov, Knight, McCoy, Miller and Solomon [4], the following result is a form of a jump inversion theorem for structures in the context of their respective degree spectra.

Theorem 1 **([4])** *Let $\mathcal{A} = (A; R)$ be a structure and for $\alpha > 1$ a computable* successor *ordinal, let $\mathcal{B}_0, \mathcal{B}_1$ be structures that satisfy the properties:*

a) $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable structures whose universes are the natural numbers and defined in the same relational language $\mathscr{L}$,
b) $\{\mathcal{B}_0, \mathcal{B}_1\}$ is α-friendly,
c) $\mathcal{B}_0, \mathcal{B}_1$ satisfy the same Σ_β sentences (of $\mathscr{L}_{\omega_1\omega}$, i.e. not only computable) for all $\beta < \alpha$,
d) each $\mathcal{B}_i$ satisfies some Σ^c_α sentence that is not true in the other.

Let $\mathcal{N}$ be the structure built as in Definition 2 for $\mathcal{A}$, $\mathcal{B}_0$ and $\mathcal{B}_1$. Then for any $X \subseteq \mathbb{N}$, $\mathcal{A}$ has a $\Delta^0_\alpha(X)$-computable copy iff $\mathcal{N}$ has an X-computable copy. It follows that

$$DS(\mathcal{A}) \subseteq \{\mathbf{a} \mid \mathbf{0}^{(\beta)} \leq \mathbf{a}\} \text{ implies } DS(\mathcal{A}) = DS_\beta(\mathcal{N}),$$

where $\beta = \alpha - 1$, if $\alpha < \omega$ and $\beta = \alpha$, if $\alpha \geq \omega$.

The proof of Theorem 1 relies on Ash's α-systems, which is a framework for priority constructions. The requirement that $\{\mathcal{B}_0, \mathcal{B}_1\}$ is α-friendly is essential for their proof.

For a set $X \subseteq \mathbb{N}$, let us denote the structure $\mathfrak{A}_X = (\mathbb{N}; X, G_S)$, where G_S is the graph of the successor function on $\mathbb{N}$. For a set $X \subseteq \mathbb{N}$ and a structure $\mathcal{A} = (A; R_0, \ldots, R_{s-1})$ with $A \cap \mathbb{N} = \emptyset$, let us denote by $A \oplus X$ the cardinal sum of the structures $\mathcal{A}$ and $\mathfrak{A}_X$, i.e. $\mathcal{A} \oplus X = (A \cup \mathbb{N}; A, \mathbb{N}, R_0, \ldots, R_{s-1}, X, G_S)$.

Our goal in this paper is to prove the following theorem, which is similar to Theorem 1, but without the requirement that $\{\mathcal{B}_0, \mathcal{B}_1\}$ is α-friendly.

Theorem 2 *Let $\mathcal{A} = (A; R)$ be a structure. Moreover, for $\alpha > 1$ a computable* successor *ordinal, let $\mathcal{B}_0, \mathcal{B}_1$ be structures that satisfy the following:*

a) $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable $\mathscr{L}$-structures whose universes are the natural numbers, where $\mathscr{L}$ is a relational language, which includes equality,
b) $\mathcal{B}_0, \mathcal{B}_1$ satisfy the same Σ^c_β sentences, for all $\beta < \alpha$,
c) each $\mathcal{B}_i$ satisfies some Σ^c_α sentence that is not true in the other.

Then for $\mathcal{N}$, built as in Definition 2 for $\mathcal{A}$, $\mathcal{B}_0$ and $\mathcal{B}_1$, we have the following:

1) $DS_\beta(\mathcal{N}) = DS(\mathcal{A} \oplus \Delta^0_\alpha)$, where $\beta = \alpha - 1$, if $\alpha < \omega$ and $\beta = \alpha$, if $\alpha \geq \omega$, and
2) $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathcal{N}) \;\leftrightarrow\; X \in \Sigma^c_1(\mathcal{A} \oplus \Delta^0_\alpha) \;\leftrightarrow\; X \in \Sigma^{c,\Delta^0_\alpha}_1(\mathcal{A})]$,

It is important to remark that the proof of Theorem 2 will not imply that if $\mathcal{A}$ has a $\Delta^0_\alpha(X)$-computable copy, then $\mathcal{N}$ has an X-computable copy. Our proof is based on the notion of forcing and building a generic copy of the structure $\mathcal{N}$.

For finite ordinals, our result can be obtained by applying a different construction, the so-called Marker's extension. It is used by A. Soskova, I. Soskov [6] and by Stukachev [7] to prove a jump inversion theorem in the context of Turing degree spectra and in the context of Σ-reducibility, respectively.

2 The notion of forcing

We define the finite parts into the set B as those finite mappings from $\mathbb{N}$ into B, which are also one-to-one. Given a finite part τ and a relation $R \subseteq B^n$, we define the finite function $\tau^{-1}(R)$ as follows:

$$\begin{aligned}\tau^{-1}(R)(u)\downarrow= 1 \leftrightarrow (\exists x_1,\dots,x_n \in \mathit{Dom}(\tau))[u = \langle x_1,\dots,x_n\rangle \ \& \\ (\tau(x_1),\dots,\tau(x_n)) \in R],\\ \tau^{-1}(R)(u)\downarrow= 0 \leftrightarrow (\exists x_1,\dots,x_n \in \mathit{Dom}(\tau))[u = \langle x_1,\dots,x_n\rangle \ \& \\ (\tau(x_1),\dots,\tau(x_n)) \notin R].\end{aligned}$$

For a structure $\mathcal{A} = (A; R_0,\dots,R_{s-1})$, we define the finite function $\tau^{-1}(\mathcal{A})$ in the following way:

1) if $u = \langle k, v\rangle$ and $k < s$, then $\tau^{-1}(\mathcal{A})(u)\downarrow= i$ iff $\tau^{-1}(R_k)(v)\downarrow= i$, for $i \in \{0,1\}$.
2) if $u = \langle k, v\rangle$, $k \geq s$, but $u < \max\{x \mid x \in \mathit{Dom}(\tau)\}$, then $\tau^{-1}(\mathcal{A})(u)\downarrow= 0$.

We remark that we need condition 2) so that we have the equality

$$f^{-1}(\mathcal{A}) = \bigcup_{\tau\subseteq f} \tau^{-1}(\mathcal{A}).$$

Partial conditions

Let us fix two structures $\mathcal{B}_0$ and $\mathcal{B}_1$ with the same universe B and in the same language $\mathscr{L}$. *Partial conditions* are finite sequences of the form

$$\mathscr{C} = (\tau^{\mathscr{C}}_0, \tau^{\mathscr{C}}_1, \dots, \tau^{\mathscr{C}}_{k-1}),$$

where every $\tau^{\mathscr{C}}_i$ is a finite part. We denote the partial conditions by the letters $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$. Let us denote the length of $\mathscr{C}$ by $|\mathscr{C}|$. For $n < |\mathscr{C}|$, we denote

$$\mathscr{C} \upharpoonright n = (\tau^{\mathscr{C}}_0, \dots, \tau^{\mathscr{C}}_{n-1}).$$

We say that $\mathscr{D}$ *extends* $\mathscr{C}$, denoted $\mathscr{C} \subseteq \mathscr{D}$, if

$$|\mathscr{C}| \leq |\mathscr{D}| \ \& \ (\forall i)[i < |\mathscr{C}| \ \rightarrow \ \tau^{\mathscr{C}}_i \subseteq \tau^{\mathscr{D}}_i].$$

We say that $\mathscr{D}$ *partially extends* $\mathscr{C}$, denoted $\mathscr{C} \subseteq_p \mathscr{D}$, if

$$|\mathscr{C}| \leq |\mathscr{D}| \;\&\; (\forall i)[i < |\mathscr{C}| \;\rightarrow\; \tau_i^{\mathscr{C}} = \tau_i^{\mathscr{D}}].$$

For a sequence of sets of natural numbers $\{B_i\}_{i<\kappa}$, with $\kappa \leq \omega$, we denote $\bigoplus_{i<\kappa} B_i = \{\langle i, x\rangle \mid i < \kappa \;\&\; x \in B_i\}$. We define the *diagram* of the partial condition $\mathscr{C}$ *with respect to* $X \in 2^\omega$ as

$$D_X(\mathscr{C}) = \bigoplus_{j<|\mathscr{C}|} (\tau_j^{\mathscr{C}})^{-1}(\mathcal{B}_{X(j)}).$$

The forcing relation

If φ is a partial function and $e \in \omega$, then by W_e^φ we denote the set of all x such that the computation $\{e\}^\varphi(x)$ halts successfully. We assume that if during a computation the oracle φ is called with an argument outside of its domain, then the computation halts unsuccessfully. Let Fin_2 be the set of all finite functions from the natural numbers taking values into $\{0,1\}$.

The definition of the forcing relation will follow the definition of the α-th Turing jump. For all natural numbers e, x, computable ordinal $\alpha \geq 1$ and partial condition $\mathscr{C}$, we define the forcing relations $\Vdash_\alpha^X$ in the following way:

(i) $\mathscr{C} \Vdash_1^X F_e(x) \;\leftrightarrow\; x \in W_e^{D_X(\mathscr{C})}$.

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned} \mathscr{C} \Vdash_{\beta+1}^X F_e(x) \;\leftrightarrow\; & (\exists \delta \in Fin_2)[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \;\&\; \mathscr{C} \Vdash_\beta^X F_z(z)) \vee \\ & (\delta(z) = 0 \;\&\; \mathscr{C} \Vdash_\beta^X \neg F_z(z))]]. \end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} \mathscr{C} \Vdash_\alpha^X F_e(x) \;\leftrightarrow\; & (\exists \delta \in Fin_2)[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \;\&\\ & ((\delta(z) = 1 \;\&\; \mathscr{C} \Vdash_{\alpha(p_z)}^X F_{x_z}(x_z)) \vee \\ & (\delta(z) = 0 \;\&\; \mathscr{C} \Vdash_{\alpha(p_z)}^X \neg F_{x_z}(x_z)))]]. \end{aligned}$$

(iv) $\mathscr{C} \Vdash_\alpha^X \neg F_e(x) \;\leftrightarrow\; (\forall \mathscr{D})[\mathscr{C} \subseteq \mathscr{D} \;\rightarrow\; \mathscr{D} \nVdash_\alpha^X F_e(x)]$.

Lemma 1. *For computable ordinals $\alpha \geq 1$ we have the following:*

1) If $\mathscr{C} \Vdash_\alpha^X F_e(x)$ and $\mathscr{C} \subseteq \mathscr{D}$, then $\mathscr{D} \Vdash_\alpha^X F_e(x)$.
2) If $\mathscr{C} \Vdash_\alpha^X \neg F_e(x)$ and $\mathscr{C} \subseteq \mathscr{D}$, then $\mathscr{D} \Vdash_\alpha^X \neg F_e(x)$.

Let δ be a finite part and $Dom(\delta) = \{d_0 < d_1 < \cdots < d_k\}$. We write $\bar{\delta}$ for the tuple $(\delta(d_0), \delta(d_1), \ldots, \delta(d_k))$. Furthermore, let us denote

$$\mathscr{C} \approx_l \mathscr{D} \;\leftrightarrow\; \bigwedge_{i \neq l} (\tau_i^{\mathscr{C}} = \tau_i^{\mathscr{D}}),$$

i.e. the partial conditions $\mathscr{C}$ and $\mathscr{D}$ are allowed to differ only in their l-th coordinates.

Note that when we say that $X \in 2^\omega$ is finite, we mean that there is i_0 such that $X(i) = 0$ for all $i > i_0$. Also, for a condition $\mathscr{C}$, we let $X_{\mathscr{C}} \in 2^\omega$ be such that $X_{\mathscr{C}}(i) = X(i)$ for $i < |\mathscr{C}|$ and $X_{\mathscr{C}}(i) = 0$ for $i \geq |\mathscr{C}|$.

Lemma 2. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable structures in the language $\mathscr{L} = \{P_0, \dots, P_{k-1}\}$, which includes equality. Let X be finite, $\mathscr{C}$ be a partial condition, l be a number such that $l < |\mathscr{C}|$, and let $D = \{x_0 < \dots < x_d\}$.*

Then for all natural numbers e, x, and a computable ordinal $\alpha \geq 1$, there is a Σ^c_α formula $\Phi^\alpha_{\mathscr{C},D,e,x}$ in $\mathscr{L}$ with free variables $X_0, \dots, X_d$ such that for every finite part ρ with $Dom(\rho) = D$, we have

$$\mathscr{D} \approx_l \mathscr{C} \ \&\ \tau^{\mathscr{D}}_l = \rho \ \&\ \mathscr{D} \Vdash^X_\alpha F_e(x) \ \leftrightarrow\ \mathcal{B}_{X(l)} \models \Phi^\alpha_{\mathscr{C},D,e,x}(\bar{\rho}).$$

We remark that if X is not computable, then $\Phi^\alpha_{\mathscr{C},D,e,x}$ will be a $\Sigma^{c,X}_\alpha$ formula.

Corollary 1. *Under the conditions of Lemma 2, for a computable ordinal $\alpha \geq 1$, there is a Σ^c_α sentence $\Phi^\alpha_{\mathscr{C},e,x}$ in the language $\mathscr{L}$ such that*

$$(\exists \mathscr{D})[\mathscr{D} \approx_l \mathscr{C} \ \&\ \mathscr{D} \Vdash^X_\alpha F_e(x)] \ \leftrightarrow\ \mathcal{B}_{X(l)} \models \Phi^\alpha_{\mathscr{C},e,x}.$$

Lemma 3. *Let us fix a computable ordinal $\alpha \geq 1$. Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable structures in the same language $\mathscr{L}$ with equality and both structures satisfy the same Σ^c_α sentences in $\mathscr{L}$. Moreover, let us fix a condition $\mathscr{C}$ and finite X, Y such that $X_{\mathscr{C}} = Y_{\mathscr{C}}$, $X \neq Y$ and they differ only at points $< m$. Then we have the equivalence:*

$$(\exists \mathscr{D} \supseteq_p \mathscr{C})[\mathscr{D} \Vdash^X_\alpha F_e(x) \ \&\ |\mathscr{D}| = m] \ \leftrightarrow\ (\exists \mathscr{D} \supseteq_p \mathscr{C})[\mathscr{D} \Vdash^Y_\alpha F_e(x) \ \&\ |\mathscr{D}| = m].$$

Proof. For $(\rightarrow)$, let us fix $\mathscr{D} \supseteq_p \mathscr{C}$ such that $\mathscr{D} \Vdash^X_\alpha F_e(x)$, $|\mathscr{D}| = m$ and let $l = |\mathscr{C}|$. For $i = l, l+1, \dots, m$, let the finite $X_i \in 2^\omega$ be such that $X_i(j) = X(j)$ for $j \notin [l, i)$ and $X_i(j) = Y(j)$ for $j \in [l, i)$. We remark that $X_l = X$ and $X_m = Y$. We shall define by induction on i the partial conditions $\mathscr{D}_i$ such that $\mathscr{D}_i \supseteq_p \mathscr{C}$, $|\mathscr{D}_i| = m$ and $\mathscr{D}_i \Vdash^{X_i}_\alpha F_e(x)$. For $i = l$, let $\mathscr{D}_i = \mathscr{D}$, which satisfies our requirements. Now suppose we have defined $\mathscr{D}_i$. Then $\mathscr{D}_i \Vdash^{X_i}_\alpha F_e(x)$ trivially implies $(\exists \mathscr{D}')[\mathscr{D}' \approx_i \mathscr{D}_i \ \&\ \mathscr{D}' \Vdash^{X_i}_\alpha F_e(x)]$. By Corollary 1, there is a Σ^c_α sentence $\Phi^\alpha_{\mathscr{D}_i,e,x}$ such that $(\exists \mathscr{D}')[\mathscr{D}' \approx_i \mathscr{D}_i \ \&\ \mathscr{D}' \Vdash^{X_i}_\alpha F_e(x)] \leftrightarrow \mathcal{B}_{X_i(i)} \models \Phi^\alpha_{\mathscr{D}_i,e,x}$. We have that $\mathcal{B}_0$ and $\mathcal{B}_1$ satisfy the same Σ^c_α sentences. Thus, $\mathcal{B}_{X_i(i)} \models \Phi^\alpha_{\mathscr{D}_i,e,x}$ iff $\mathcal{B}_{Y(i)} \models \Phi^\alpha_{\mathscr{D}_i,e,x}$. Since $X_{i+1}(i) = Y(i)$ and $X_i(j) = X_{i+1}(j)$ for $j \neq i$, by Corollary 1, $(\exists \mathscr{D}')[\mathscr{D}' \approx_i \mathscr{D}_i \ \&\ \mathscr{D}' \Vdash^{X_{i+1}}_\alpha F_e(x)] \leftrightarrow \mathcal{B}_{Y(i)} \models \Phi^\alpha_{\mathscr{D}_i,e,x}$. By combining the above equivalences, we obtain

$$(\exists \mathscr{D}')[\mathscr{D}' \approx_i \mathscr{D}_i \ \&\ \mathscr{D}' \Vdash^{X_i}_\alpha F_e(x)] \ \leftrightarrow\ (\exists \mathscr{D}')[\mathscr{D}' \approx_i \mathscr{D}_i \ \&\ \mathscr{D}' \Vdash^{X_{i+1}}_\alpha F_e(x)].$$

We set $\mathscr{D}_{i+1}$ to be this $\mathscr{D}' \approx_i \mathscr{D}_i$ such that $\mathscr{D}' \Vdash^{X_{i+1}}_\alpha F_e(x)$. Since $i \geq |\mathscr{C}| = l$ and $\mathscr{D}_i \supseteq_p \mathscr{C}$, we have $\mathscr{D}_{i+1} \supseteq_p \mathscr{C}$. Eventually, we obtain $\mathscr{D}_m$ such that $|\mathscr{D}_m| = m$, $\mathscr{D}_m \supseteq_p \mathscr{C}$ and $\mathscr{D}_m \Vdash^Y_\alpha F_e(x)$. The direction $(\leftarrow)$ is symmetric. $\square$

Lemma 4. *Let us fix a computable ordinal $\alpha \geq 1$. Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable structures in the language $\mathscr{L}$ with equality and both structures satisfy the same Σ^c_α sentences in $\mathscr{L}$. Then for every partial condition $\mathscr{C}$, $X \in 2^\omega$ and natural numbers e, x:*

1) $\mathscr{C} \Vdash^X_\alpha F_e(x) \;\leftrightarrow\; \mathscr{C} \Vdash^{X_\mathscr{C}}_\alpha F_e(x)$,
2) $\mathscr{C} \Vdash^X_\alpha \neg F_e(x) \;\leftrightarrow\; \mathscr{C} \Vdash^{X_\mathscr{C}}_\alpha \neg F_e(x)$.

Proof. We prove 1) and 2) simultaneously by transfinite induction on α.

Let $\alpha = 1$. For 1), it is clear, by the definition of $\Vdash^X_1$, that for every e and x,

$$\mathscr{C} \Vdash^X_1 F_e(x) \;\leftrightarrow\; \mathscr{C} \Vdash^{X_\mathscr{C}}_1 F_e(x).$$

For 2), we have two cases to consider.

i) Let $\mathscr{C} \Vdash^X_1 \neg F_e(x)$ and assume $\mathscr{C} \not\Vdash^{X_\mathscr{C}}_1 \neg F_e(x)$. Fix $\mathscr{D}_0 \supseteq \mathscr{C}$ such that $\mathscr{D}_0 \Vdash^{X_\mathscr{C}}_1 F_e(x)$ and let $m = |\mathscr{D}_0|$, $\mathscr{D}' = \mathscr{D}_0 \restriction |\mathscr{C}|$. Since $X_\mathscr{C} = X_{\mathscr{D}'}$, we have $(\exists \mathscr{D} \supseteq_p \mathscr{D}')[\mathscr{D} \Vdash^{X_{\mathscr{D}'}}_1 F_e(x) \;\&\; |\mathscr{D}| = m]$. Since the finite $X_{\mathscr{D}_0}, X_{\mathscr{D}'}$ differ only at positions $< m$ and $(X_{\mathscr{D}_0})_{\mathscr{D}'} = X_{\mathscr{D}'}$, by Lemma 3, $(\exists \mathscr{D} \supseteq_p \mathscr{D}')[\mathscr{D} \Vdash^{X_{\mathscr{D}_0}}_1 F_e(x) \;\&\; |\mathscr{D}| = m]$. We conclude that there is $\mathscr{D} \supseteq_p \mathscr{D}' \supseteq \mathscr{C}$ such that $\mathscr{D} \Vdash^{X_\mathscr{D}}_1 F_e(x)$ and by 1), $\mathscr{D} \Vdash^X_1 F_e(x)$. We reach a contradiction with $\mathscr{C} \Vdash^X_1 \neg F_e(x)$.

ii) Let $\mathscr{C} \Vdash^{X_\mathscr{C}}_1 \neg F_e(x)$ and assume $\mathscr{C} \not\Vdash^X_1 \neg F_e(x)$. In a similar way as in i) we show that we can apply Lemma 3 to reach a contradiction with $\mathscr{C} \Vdash^{X_\mathscr{C}}_1 \neg F_e(x)$.

For $\alpha > 1$, case 1) follows easily by the definition of the forcing relation $\Vdash^X_\alpha$ and the induction hypothesis for cases 1) and 2). Since we can apply Lemma 3 for every $\beta \leq \alpha$, the proof of 2) for $\alpha > 1$ is essentially the same as for $\alpha = 1$. ◻

Total conditions

Let us again fix structures $\mathcal{B}_0$ and $\mathcal{B}_1$ with the same universe B. The *total conditions* are infinite sequences $\mathbf{C} = (f_0, f_1, f_2, \dots, f_i, \dots)$, where for all i, f_i is an enumeration of the set B. We denote the total conditions by the letters $\mathbf{C}$ and $\mathbf{G}$. We define the *diagram* of $\mathbf{C}$ *with respect* to $X \in 2^\omega$ to be

$$D_X(\mathbf{C}) = \bigoplus_{j<\omega} f_j^{-1}(\mathcal{B}_{X(j)}).$$

For total conditions, we define the modelling relation $\models^X_\alpha$ for every computable ordinal $\alpha \geq 1$ in a way that mirrors the definition of the forcing relation:

(i) $\mathbf{C} \models^X_1 F_e(x) \;\leftrightarrow\; x \in W_e^{D_X(\mathbf{C})}$
(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned}\mathbf{C} \models^X_{\beta+1} F_e(x) \;\leftrightarrow\; & (\exists \delta \in Fin_2)[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \;\&\; \mathbf{C} \models^X_\beta F_z(z)) \vee \\ & (\delta(z) = 0 \;\&\; \mathbf{C} \models^X_\beta \neg F_z(z))]].\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned}\mathbf{C} \models^X_\alpha F_e(x) \ \leftrightarrow\ & (\exists \delta \in Fin_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \&\\ & ((\delta(z) = 1 \ \&\ \mathbf{C} \models^X_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\ & (\delta(z) = 0 \ \&\ \mathbf{C} \models^X_{\alpha(p_z)} \neg F_{x_z}(x_z)))]].\end{aligned}$$

(iv) $\mathbf{C} \models^X_\alpha \neg F_e(x) \ \leftrightarrow\ \mathbf{C} \not\models^X_\alpha F_e(x)$.

Lemma 5. *Let* $\mathbf{C}$ *be a total condition and* $\alpha \geq 1$ *be a computable ordinal. Then*

$$x \in W_e^{\Delta^0_\alpha(D_X(\mathbf{C}))} \ \leftrightarrow\ \mathbf{C} \models^X_\alpha F_e(x).$$

For a computable ordinal $\alpha \geq 1$, we say that $\mathbf{C}$ is α-*generic* with respect to X if for every e, x and $1 \leq \beta < \alpha$, $(\exists \mathscr{C} \subset \mathbf{C})[\mathscr{C} \Vdash^X_\beta F_e(x) \vee \mathscr{C} \Vdash^X_\beta \neg F_e(x)]$.

Lemma 6. *For every computable ordinal* $\alpha \geq 1$ *we have the following:*

1) Let $\mathbf{C}$ *be* α*-generic with respect to* X*. Then*

$$\mathbf{C} \models^X_\alpha F_e(x) \ \leftrightarrow\ (\exists \mathscr{C} \subset \mathbf{C})[\mathscr{C} \Vdash^X_\alpha F_e(x)].$$

2) Let $\mathbf{C}$ *be* $(\alpha+1)$*-generic with respect to* X*. Then*

$$\mathbf{C} \models^X_\alpha \neg F_e(x) \ \leftrightarrow\ (\exists \mathscr{C} \subset \mathbf{C})[\mathscr{C} \Vdash^X_\alpha \neg F_e(x)].$$

3 Construction of a generic copy of $\mathcal{N}$

For two functions f and h, let us denote $E(f,h) = \{\langle x, y\rangle \mid f(x) = h(y)\}$.

Proposition 1 *Let* $\mathcal{A} = (A; R)$*,* $R \subseteq A^n$*, and* $\mathcal{N}$ *be defined as in Definition 2. For every total condition* $\mathbf{C} = (q_0, q_1, \dots)$ *and total enumeration* f *of* $\mathcal{A}$*, there is an enumeration* $h_{\mathbf{C}}$ *of* $\mathcal{N}$ *such that* $h^{-1}_{\mathbf{C}}(\mathcal{N}) \leq_T D_{f^{-1}(R)}(\mathbf{C})$ *and* $E(h_{\mathbf{C}}, f)$ *is computable.*

Proposition 2 *For every enumeration* f *of* $\mathcal{A} \oplus X$*, there is a total enumeration* h *of* $\mathcal{A}$ *such that*

1) $E(f,h) \leq_T f^{-1}(\mathcal{A} \oplus X)$*, and*
2) $h^{-1}(\mathcal{A}) \oplus X \leq_T f^{-1}(\mathcal{A} \oplus X)$*.*

Lemma 7. *Let* $\mathcal{A} = (A; R)$*,* α *be a computable* successor *ordinal, and* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *be computable structures such that:*

a) $\mathcal{B}_0, \mathcal{B}_1$ *are defined in the same language* $\mathscr{L}$*, which includes equality,*
b) $\mathcal{B}_0, \mathcal{B}_1$ *satisfy the same* Σ^c_β *sentences in* $\mathscr{L}$ *for all* $\beta < \alpha$*.*

Then for every enumeration f *of* $\mathcal{A} \oplus \Delta^0_\alpha$*, there is an enumeration* g *of* $\mathcal{N}$ *such that*

1) $E(f,g) \leq_T f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$,
2) $\Delta^0_\alpha(g^{-1}(\mathcal{N})) \leq_T f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$.

Proof. Let $\alpha = \beta + 1$. By Proposition 2, let us fix, for the given enumeration f of $\mathcal{A} \oplus \Delta^0_\alpha$, a total enumeration h of $\mathcal{A}$ such that $h^{-1}(\mathcal{A}) \oplus \Delta^0_\alpha \leq f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$ and $E(f,h)$ is computable. Our goal is to build a total α-generic condition $\mathbf{G}$ in stages, such that $\mathbf{G} = \bigcup \mathscr{C}_k$. The desired enumeration g will be $h_{\mathbf{G}}$, defined as in Proposition 1. At each stage k, we define a partial condition $\mathscr{C}_{k+1}$ and a finite $X_{k+1} \in 2^\omega$ such that $X_{k+1} = h^{-1}(R) \restriction |\mathscr{C}_{k+1}|$. Let $\mathscr{C}_0 = \emptyset$ and $X_0 = \emptyset$. At step $k = \langle e, x\rangle + 1$, we ask whether $(\exists \mathscr{D} \supseteq \mathscr{C}_k)[\mathscr{D} \Vdash^{X_k}_\beta F_e(x)]$. Since X_k is finite and $\mathcal{B}_0, \mathcal{B}_1$ are computable, this question can be expressed by a Σ^c_β sentence and thus we can decide whether such $\mathscr{D}$ exists effectively relative to Δ^0_α.

If such $\mathscr{D}$ does not exists, then, by definition, $\mathscr{C}_k \Vdash^{X_k}_\beta \neg F_e(x)$. We set $\mathscr{C}_{k+1} = \mathscr{C}_k$, $X_{k+1} = X_k$ and go to the next step.

If such $\mathscr{D}$ exists, let $\mathscr{E} = \mathscr{D} \restriction |\mathscr{C}_k|$ and $X' = h^{-1}(R) \restriction |\mathscr{D}|$. Since $X_k = h^{-1}(R) \restriction |\mathscr{C}_k|$, we have $(X')_{\mathscr{E}} = X_k$. Then according to Lemma 3, $(\exists \mathscr{D}' \supseteq_p \mathscr{E})[\mathscr{D}' \Vdash^{X'}_\alpha F_e(x)\ \&\ |\mathscr{D}'| = |\mathscr{D}|]$. We can find the pair $(\mathscr{D}', X')$ such that $\mathscr{D}' \Vdash^{X'}_\alpha F_e(x)$ effectively relative to $h^{-1}(\mathcal{A}) \oplus \Delta^0_\alpha$. Then, if necessary, we enlarge $\mathscr{D}'$ so that for every $i < |\mathscr{C}_k|$, $\tau^{\mathscr{D}'}_i$ is defined on an initial segment of $\mathbb{N}$ and $\tau^{\mathscr{D}'}_i \supsetneq \tau^{\mathscr{C}_k}_i$. By the monotonicity property of the forcing relation, that is Lemma 1, we know that we can do this safely. We set $\mathscr{C}_{k+1}$ to be this enlarged $\mathscr{D}'$ and set $X_{k+1} = X'$. Then we go to the next step.

In the end, we set $\mathbf{G} = \bigcup_i \mathscr{C}_i$, where $g_k = \bigcup_i \tau^{\mathscr{C}_i}_k$ and $\mathbf{G} = (g_0, g_1, \dots)$. By Proposition 1, for $\mathbf{G}$ we define the enumeration $h_{\mathbf{G}}$ of $\mathcal{N}$. Then

$$\begin{aligned} x \in \Delta^0_\alpha(h^{-1}_{\mathbf{G}}(\mathcal{N})) &\leftrightarrow \mathbf{G} \models^{h^{-1}(R)}_\beta F_{\mu(x,\beta)}(x) \\ &\leftrightarrow (\exists k)[\mathscr{C}_k \subseteq \mathbf{G}\ \&\ \mathscr{C}_k \Vdash^{h^{-1}(R)}_\beta F_{\mu(x,\beta)}(x)] \\ &\leftrightarrow (\exists k)[\mathscr{C}_k \subseteq \mathbf{G}\ \&\ \mathscr{C}_k \Vdash^{X_k}_\beta F_{\mu(x,\beta)}(x)]. \end{aligned}$$

By the construction above, we know that at step $k = \langle \mu(x,\beta), x\rangle + 1$ we have answered the question whether $\mathscr{C}_k \Vdash^{X_k}_\beta F_{\mu(x,\beta)}(x)$ or $\mathscr{C}_k \Vdash^{X_k}_\beta \neg F_{\mu(x,\beta)}(x)$. Since the sequence $\{\langle \mathscr{C}_k, X_k\rangle\}_{k\in\omega}$ is computable in $h^{-1}(\mathcal{A}) \oplus \Delta^0_\alpha$, we conclude that $\Delta^0_\alpha(h^{-1}_{\mathbf{G}}(\mathcal{N})) \leq_T h^{-1}(\mathcal{A}) \oplus \Delta^0_\alpha \leq_T f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$. Moreover, by Proposition 1, $E(h_{\mathbf{G}}, h)$ is computable and since $E(h,f) \leq_T f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$ it follows that $E(h_{\mathbf{G}}, f) \leq_T f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$. □

Corollary 2. *Under the conditions of Lemma 7, we have the following:*

1) $DS(\mathcal{A} \oplus \Delta^0_\alpha) \subseteq DS_\beta(\mathcal{N})$, *where* $\beta = \alpha - 1$, *if* $\alpha < \omega$ *and* $\beta = \alpha$, *if* $\alpha \geq \omega$*;*
2) $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathcal{N}) \ \rightarrow\ X \in \Sigma^c_1(\mathcal{A} \oplus \Delta^0_\alpha)]$.

Proof. We proved in Lemma 7 that for every enumeration f of the structure $\mathcal{A} \oplus \Delta^0_\alpha$, there is an enumeration h of $\mathcal{N}$ such that $\Delta^0_\alpha(h^{-1}(\mathcal{N})) \leq_T f^{-1}(\mathcal{A} \oplus \Delta^0_\alpha)$. Then Property 1) follows from the fact that the degree spectra of $\mathcal{A} \oplus \Delta^0_\alpha$ and $\mathcal{N}$ are closed upwards.

Property 2) follows easily from the theorem by Ash-Knight-Manasse-Slaman [2] and Chisholm [3] that the relatively intrinsically Σ^0_α relations in a structure $\mathcal{A}$ are exactly the Σ^c_α definable relations in $\mathcal{A}$. □

Lemma 8. *Let $\mathcal{A} = (A; R)$, α be a computable* successor *ordinal and $\mathcal{B}_0, \mathcal{B}_1$ be computable structures such that:*

a) $\mathcal{B}_0, \mathcal{B}_1$ are defined in the same language $\mathscr{L}$, which includes equality,
b) each $\mathcal{B}_i$ satisfies some Σ^c_α sentence in $\mathscr{L}$ that is not true in the other.

Then for every enumeration f of $\mathcal{N}$, there is an enumeration h of $\mathcal{A} \oplus \Delta^0_\alpha$ such that:

1) $E(f, h) \leq_T f^{-1}(\mathcal{N})$, and
2) $h^{-1}(\mathcal{A} \oplus \Delta^0_\alpha) \leq_T \Delta^0_\alpha(f^{-1}(\mathcal{N}))$.

Proof. Let f be the given enumeration of $\mathcal{N}$. We define h, an enumeration of $A \cup \mathbb{N}$, as $h(2n) = f(n)$ for all $n \in f^{-1}(A)$ and $h(2n+1) = n$, for all $n \in \mathbb{N}$. It is clear that $E(f, h)$ is computable in $f^{-1}(\mathcal{N})$.

For any $x_1, \ldots, x_n$, let $i = \langle x_1, \ldots, x_n \rangle$ and $\bar{a}_i = (f(x_1), \ldots, f(x_n))$. To check if $2i \in h^{-1}(R)$, we need to determine k in $\mathcal{U}_{\bar{a}_i} \cong \mathcal{B}_k$. Since we have Σ^c_α sentences Φ and Ψ such that $\mathcal{B}_0 \models (\Phi \ \& \ \neg\Psi)$ and $\mathcal{B}_1 \models (\neg\Phi \ \& \ \Psi)$, we can do that effectively relative to $\Delta^0_\alpha(f^{-1}(\mathcal{N}))$. Thus, $h^{-1}(R) \leq_T \Delta^0_\alpha(f^{-1}(\mathcal{N}))$.

The sets $h^{-1}(G_S)$ and $h^{-1}(\mathbb{N})$ are computable and since $h^{-1}(\Delta^0_\alpha) \equiv_T \Delta^0_\alpha$, we conclude that $h^{-1}(\mathcal{A} \oplus \Delta^0_\alpha) \leq_T \Delta^0_\alpha(f^{-1}(\mathcal{N}))$. □

We conclude by stating the following corollary, which is symmetric to Corollary 2.

Corollary 3. *Under the conditions of Lemma 8, we have the following:*

1) $DS_\beta(\mathcal{N}) \subseteq DS(\mathcal{A} \oplus \Delta^0_\alpha)$, where $\beta = \alpha - 1$, if $\alpha < \omega$ and $\beta = \alpha$, if $\alpha \geq \omega$;
2) $(\forall X \subseteq A)[X \in \Sigma^c_1(\mathcal{A} \oplus \Delta^0_\alpha) \ \rightarrow \ X \in \Sigma^c_\alpha(\mathcal{N})]$.

Now Corollary 2 and Corollary 3 gives us exactly Theorem 2.

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