A MINIMAL VOLUME ELLIPSOID
AROUND A SIMPLEX

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This paper presents a complete proof of the fact \(^1\) that the ellipsoid with minimal volume containing the regular simplex is a sphere (lemma 4). It is shown also that the proof can be used to build and compute easily minimal volume ellipsoids around irregular simplex (Theorem 1). Some consequences are presented.

The first two lemmas deal with the well-known (see [2],[3]) Steiner and Swartz symmetry mappings applied to ellipsoids.

Let \( \mathbb{R}^p \) be the \( p \)-dimensional euclidean space.

**Lemma 1** The Steiner symmetry mapping for any ellipsoid may be extended to an affine transformation defined on the whole space.

**Proof.** Consider the ellipsoid

\[
E(\mu, \Sigma) = \{ x : (x - \mu)^T \Sigma (x - \mu) \leq 1 \},
\]

where \( \Sigma \) is positive definite matrix. We may suppose first that \( \mu = 0 \) and the hyperplane of symmetry is defined by \( x_1 = 0 \). Let fix a point \( P \) in this hyperplane with coordinates \( 0, x_2, x_3, \ldots x_p \). The Steiner mapping works in the following way. One have:

- to solve the equation in (1) according to the first coordinate \( x_1 \),

\(^1\)The statement itself seems to be wellknown, however we could not find an explicit proof
to find the intersection points \(P_1\) and \(P_2\) (with the same coordinates as \(P\) except the first one) on the surface of \(E\),

- to find the middle point \(M(P) = (P_1 + P_2)/2\).

The Steiner mapping translates the interval \([P_1, P_2]\), so that \(M(P)\) coincide with \(P\). The translation acts on the first coordinate only. The main point is that \(M(P)\) depends linearly on \(P\).

Denote by \(Q\) the projector operator on the first coordinate. We will define the Steiner mapping in the following way:

\[
S(x) = (I + c.Q\Sigma(I - Q))x,
\]

with \(c = 1/(2z_{11})\). It is easy to check that after this mapping the point \(M(P)\) goes to \(P\). As \(\det(S) = 1\), the resulting ellipsoid \(E(0, \Sigma')\) will have the same volume and will be described by the following matrix \(\Sigma'\):

\[
z'_{11} = \frac{\det(\Sigma)}{\det(\Sigma')}, \quad z'_{ij} = \delta_{i1} = 0,
\]

\[
z''_{i-1,j-1} = z'_{ij}, \quad i, j = 2, 3, \ldots p,
\]

where with \(\Sigma''\) we denote the the main minor (submatrix) of \(\Sigma\).

Thus we have constructed a linear equivalent of the Steiner symmetrisation in the case \(\mu = 0.\) The extension to affine transformation in the general case is obvious.

**Lemma 2** Let \(E\) be an ellipse in \(\mathbb{R}^p\) \((p = 2)\). Let it be invariant under two different and non-orthogonal Steiner symmetries. Then \(E\) is a circle.

**Proof.** Without loss of generality we may suppose that \(\mu = 0.\) Consider now equation in (1): \(x^t\Sigma x = 1.\) According the first symmetry (the line of symmetry is \(x = 0\), for example) (1) may be written in the form:

\[
ax^2 + by^2 = 1, \quad a, b > 0.
\]

(2)

Let the angle between first and second direction of symmetrisation be \(\theta.\) Denote by \(R(\theta)\) the corresponding rotation matrix. Then the stability according the second Steiner mapping implies: \(R(\theta)\Sigma R'(\theta) = \Sigma.\) This leads to the following equation for the element out of the diagonal:

\[
(a - b)\sin(\theta)\cos(\theta) = 0.
\]
As $\theta$ is not equal to $(k\pi)/2$ for any $k$, it follows that $a = b$. ■

In [2] pp.106-112 the rotation symmetry of Swarz is extracted from the Steiner symmetrisation: the invariance under two Steiner mappings imply invariance with respect to the rotation around the intersection line of corresponding hyperplanes. In the case of ellipsoids the restriction for the angle $\theta$ between the corresponding hyperplanes $'\theta = (k\pi)/m'$ is weakened to $'\theta = (k\pi)/2'$. Following two lemma’s are devoted to the simplex in $\mathbb{R}^p$. Any simplex $X$ will be identified with the $(p$ by $p+1)$ matrix (denoted with the same letter $X$) which contains the coordinates of its vertices as columns. So $\text{vol}(X)$ is the volume of the convex set spanned by the points and $\text{rad}(Y)$ is the circumradius of the sphere with center in the barycentre of these points.

**Lemma 3** Consider the standard simplex $Y$ in $\mathbb{R}^p$ with distance between vertices equal to 1. Then we have the following formulas:

\[
\text{vol}(Y) = (p+1)^{3/2} \frac{2^{-p/2}}{p}, \quad (3)
\]
\[
\text{rad}(Y) = (p/2)^{1/2} (p+1)^{-1/2}. \quad (4)
\]

**Proof.** Denote by $A$ the $(p$ by $p+1)$ matrix defined in the following way:

\[
a_{i,j} = \begin{cases} 
-2^{-1/2} \frac{1}{(i \ast (i + 1))^{1/2}}, & j = 0, 1, \ldots, i - 1, \\
-2^{-1/2} \frac{1}{(i/(i + 1))^{1/2}}, & i = j, \\
0, & i < j.
\end{cases}
\]

Denote the columns of $A$ by $y_j$, $j = 0, 1, \ldots, p$. It is easy to see that these are the vertices of a standard centered simplex in $\mathbb{R}^p$. In fact we have to check the following relations:

\[
||y_j - y_i|| = \delta_{ij}, \sum y_j = 0, ||y_j|| = ||y_p|| = a_{pp}, j = 0, 1, \ldots, p.
\]

Thus (4) is proved. Denote now by $B$ the right $(p$ by $p)$ submatrix of $A$. Since the matrix $B$ is lower triangle we have

\[
\text{det}(B) = (p + 1)^{1/2} 2^{-p/2}.
\]

Since $Y$ consists of exactly $p + 1$ subsimplexes with vertex in 0 and volume equal to $\text{det}(B)/p$, we have:

\[
\text{vol}(Y) = (p + 1)^{3/2} \frac{2^{-p/2} \cdot p^{-1}}{p}. ■
\]
Lemma 4  The minimal volume ellipsoid $E$ containing the standard simplex $Y$ is a sphere with center in the origin.

Proof.  The proof is carried out by using the Steiner mapping defined in Lemma 1. Suppose that the standard simplex $Y$ is the same as above. Denote by $E$ the minimal volume ellipsoid containing it.

Denote by $P_1$ and $P_2$ any two vertices of it and by $L$ the hyperplane which is normal to the vector $P_1 - P_2$ and crosses it in the point $(P_1 + P_2)/2$. It is easy to check that $L$ contains the point 0 and all the remaining vertices of $Y$. Now we will build the Steiner mapping $S$ for $E$ on $L$ and prove that it is in fact identity.

Denote by $S(E)$ the symmetrized ellipsoid. We have that $S(P_1) = P_1, S(P_2) = P_2$. Suppose that for some vertex $P_3$ $S(P_3)$ is not equal to $P_3$. This means that $S(P_3)$ is outside $L$.

Let us consider the two dimensional plane $H$ containing $P_1, P_2$ and $P_3$. As any point is moved along the direction $P_2 - P_1$, the point $S(P_3)$ also belong to it. The set $H \cap S(E)$ is an ellipse and contains $P_1, P_2$ and $S(P_3)$. From the symmetry $H$ contains also $P_3 - (S(P_3) - P_3)$. From the strict convexity it follows that the point $P_3$ belongs to the open interior of $H \cap S(E)$ and to the open interior $S(E)$. This contradicts to the fact that $E$ was a minimal volume ellipsoid.

Thus the ellipsoid $E$ is symmetric in any direction $P_j - P_i$ and following Swarz symmetry principle (see Lemma 2), it is a sphere. □

Consider now an arbitrary simplex $X$ (a $(p \times p + 1$ - matrix), defined by $(p + 1)$ vertices (columns) in $E$.

Theorem 1  There exists ellipsoid $E$ with minimal volume, such that it contains $X$. $E$ is centered in the barycentre of $X$. The volume of $E$ may be computed by the following formula:

$$\text{vol}(E) = K_p \text{vol}(X),$$

$$K_p = \pi^{n/2} p^{(p+2)/2} (p + 1)^{-(p+3)/2} / \Gamma(1 + p/2).$$

Proof.  Let us center the vertices of $X$ by subtracting the barycentre and solve the linear system $X = LY$ according to $L$.

Denote the rightmost square submatrix of centred $X$ by $A$. Consider the matrix $B$ the same (as in Lemma 3) submatrix of the standard simplex $Y$. The solution of the linear system $A = LB$ according $L$ is clearly the same.
Since $B$ is nonsingular the solution exists and is unique. We have $\det(L) = \det(A)/\det(B)$. According to Lemma 4 the minimal volume ellipsoid around $Y$ is a sphere $S$. Translated to $X$ using linear mapping $L$ it will again be a minimal volume ellipsoid. The volume of a sphere in $E$ with radius $r$ is equal to: $r^p \pi^{p/2} / \Gamma(1 + p/2)$. ($\Gamma(.)$ is the gamma function). Simple computations give (4):

$$\text{vol}(L.S) = \text{vol}(S) \cdot \det(L) = \text{vol}(S) \cdot \det(X) / \det(Y) =$$

$$= \left(\text{vol}(S) / \text{vol}(Y)\right) \cdot \text{vol}(X) = K_p \cdot \text{vol}(X).$$

Here for simplicity we use $\det(X)$ instead of $\det(A)$ and $\det(Y)$ instead of $\det(B)$. ■

**Corollary 1.** The maximum volume simplex inscribed in an ellipsoid $E$ has the following properties:

1. Its barycenter coincides with the center of $E$.
2. Its volume is computed according (4).
3. Any point from the surface of $E$ may be vertex of such simplex.

**Corollary 2.** 1. The computation of the minimal volume circumscribed ellipsoid around a symplex follows easily from the following simple relations. Let $Y$ be the standard centred symplex with normed vertices and $e$ be normed vector with equal coordinates in $\mathbb{R}^{p+1}$. We have

$$Y'.Y = c(I - e.e'), \quad c = (p + 1)/p.$$  

For any nondegenerate symplex $X$ we have:

$$L.X = Y, \quad X.e = 0 = Y.e = 0.$$  

Then for unknown $\Sigma$ we easily calculate:

$$X'.L'.L.X = c(I - e.e'), \quad \Sigma = L'.L,$$

$$X.X'.L'.L.X' = c.X.X',$$

$$\Sigma = c(X.X')^{-1}.$$
2. From the same line of computations follows the uniqueness of the circumscribed ellipsoid.

Finally we would like to mention that the circumscribed simplex appeared in different fields of applications, i.e. robust statistics (see [4]) and the solution may be be useful.

References


