

GIBBS VECTORS IN THREE-DIMENSIONAL VECTOR SPACES OVER ORDERED FIELDS

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Non nova, sed nove

In the present paper the symbols Sgn, sgn:, Ax, Df, Pr and Dm stand for the words notation, denote, axiom, definition, proposition and proof respectively.

F stands for an arbitrary non-trivial ordered field and V_F for the three-dimensional vector space over F , defined as a set, for which mappings

$$1) \quad M_1: V_F^2 \rightarrow V_F$$

(addition in V_F),

$$2) \quad M_2: F \times V_F \rightarrow V_F$$

(multiplication of the elements of F with the elements of V_F),

$$3) \quad M_3: V_F^2 \rightarrow F$$

(scalar multiplication of the elements of V_F) and

$$4) \quad M_4: V_F^2 \rightarrow V_F$$

(vector multiplication in V_F) are defined, so that, provided

$$5) \quad a+b \text{ sgn: } M_1((a, b))$$

(sum of a, b),

$$6) \quad \lambda a \text{ sgn: } M_2((\lambda, a))$$

(product of λ, a),

$$7) \quad ab \text{ sgn: } M_3((a, b))$$

(scalar product of a, b) and

$$8) \quad a \times b \text{ sgn: } M_4((a, b))$$

(vector product of a, b), the following conditions are satisfied:

Ax 1. $a, b, c \in V_F$ imply $(a+b)+c=a+(b+c)$.

Ax 2. There exists $0 \in V_F$ with: $a \in V_F$ implies $a+0=a$.

Ax 3. $a \in V_F$ implies: there exists $-a \in V_F$ with $a+(-a)=0$.

- Ax 4.** $a \in V_F$ implies $1a = a$.
- Ax 5.** $\lambda, \mu \in F$, $a \in V_F$ imply $(\lambda\mu)a = \lambda(\mu a)$.
- Ax 6.** $\lambda, \mu \in F$, $a \in V_F$ imply $(\lambda + \mu)a = \lambda a + \mu a$.
- Ax 7.** $\lambda \in F$; $a, b \in V_F$ imply $\lambda(a+b) = \lambda a + \lambda b$.
- Ax 8.** $a, b \in V_F$ imply $ab = ba$.
- Ax 9.** $\lambda \in F$; $a, b \in V_F$ imply $(\lambda a)b = \lambda(ab)$.
- Ax 10.** $a, b, c \in V_F$ imply $(a+b)c = ac + bc$.
- Ax 11.** $a \in V_F$ implies $0 \leq a^2$.
- Ax 12.** $a \in V_F$, $a^2 = 0$ imply $a = 0$.
- Ax 13.** $a, b \in V_F$ imply $a \times b = -(b \times a)$.
- Ax 14.** $\lambda \in F$; $a, b \in V_F$ imply $(\lambda a) \times b = \lambda(a \times b)$.
- Ax 15.** $a, b, c \in V_F$ imply $(a+b) \times c = a \times c + b \times c$.
- Ax 16.** $a, b, c \in V_F$ imply $a \times b \cdot c = a \cdot b \times c$.
- Ax 17.** $a, b, c \in V_F$ imply $(a \times b) \times c = (ac)b - (bc)a$.
- Ax 18.** There exist $a, b \in V_F$ with $a \times b \neq 0$.

0 in Ax 11, Ax 12 and 1 in Ax 4 denote the zero and the unit element of F respectively and $a - b$ sgn: $a + (-b)$ for all a and b of V_F (difference of a, b); a^2 sgn: aa for every $a \in V_F$.

The conditions ax 1 — Ax 18 are called axioms for three-dimensional vector space V_F over F .

Ax 1 — Ax 3 specify V_F to an additive group with respect to M_1 with zero element O and inverse element $-a$ for the element a of V_F ; it is proved, that this group is commutative.

Ax 1 — Ax 7 specify V_F to a linear space over F with respect to M_1 and M_2 .

Ax 1 — Ax 12 specify V_F to an Euclidean space over F with respect to M_1 , M_2 and M_3 .

It is proved that every four elements of V_F are linearly dependent and that if a and b are linearly independent elements of V_F the elements a , b and $a \times b$ of V_F are linearly independent also. Now Ax 18 eliminates the one-dimensional case and turns V_F into a three-dimensional linear space over F .

The consistency of the system of axioms Ax 1 — Ax 18 is traditionally proved by constructing an arithmetical model of V_F in F^3 : (5) — (8) are defined as follows:

$$(9) \quad (a_1, a_2, a_3) + (b_1, b_2, b_3) \text{ sgn: } (a_1 + b_1, a_2 + b_2, a_3 + b_3),$$

$$(10) \quad \lambda(a_1, a_2, a_3) \text{ sgn: } (\lambda a_1, \lambda a_2, \lambda a_3),$$

$$(11) \quad (a_1, a_2, a_3)(b_1, b_2, b_3) \text{ sgn: } \sum_{s=1}^3 a_s b_s,$$

$$(12) \quad (a_1, a_2, a_3) \times (b_1, b_2, b_3) \text{ sgn: }$$

$$(a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2)$$

provided

$$(13) \quad \lambda \in F,$$

$$(14) \quad (a_1, a_2, a_3) \in F^3,$$

$$(15) \quad (b_1, b_2, b_3) \in F^3$$

and Ax 1 — Ax 18 are verified; it is proved that V_F is a three-dimensional linear space over F with respect to (9), (10) and this implies the categoricity of the system of axioms Ax 1 — Ax 18.

The question to what degree the mathematical theory involved depends on the arithmetical specificity of the field F quite naturally arises: it is immediately seen, that all the axioms Ax 4 — Ax 12, Ax 14, Ax 16 and Ax 17 remain meaningful if F is replaced by an arbitrary additive-multiplicative bistructure B with zero element 0 and unity element 1, in which an order relation $<$ is defined compatible with the addition and multiplication in B . In case the system of axioms Ax 1 — Ax 18, properly modified, remains consistent, the mathematical theory so established is of course void of content: it becomes sapid when various expedient properties to the relations addition and multiplication in B are attributed.

Systematical investigations on the interaction of the various axioms of the system Ax 1 — Ax 18 are still lacking: we do not have at our disposal for example proofs for the independence of these axioms; but these questions for the time being will be left aside. At any rate the condition F to be an ordered field is an essential one, as may be seen from the example with the field C of the complex numbers.

Namely, it is easily seen that if L_C is a linear space over C with two at least linearly independent elements p and q it is impossible to define scalar multiplication of the elements of L_C with values in C which satisfies the following conditions:

Ax 8C. $a, b \in L_C$ imply $ab = ba$.

Ax 9C. $\lambda \in C; a, b \in L_C$ imply $(\lambda a) b = \lambda (a b)$.

Ax 10C. $a, b, c \in L_C$ imply $(a+b)c = ac + bc$.

Ax 12C. $a \in L_C, a^2 = 0$ imply $a = 0$.

To this end it is sufficient to consider the quadratic equation

$$(16) \quad \lambda^2 p^2 + 2\lambda pq + q^2 = 0$$

with roots obviously in C ; Ax 8C — Ax 10C imply

$$(17) \quad (\lambda p + q)^2 = \lambda^2 p^2 + 2\lambda pq + q^2 = 0$$

contrary to Ax 12C since

$$(18) \quad \lambda p + q = \lambda p + 1 q \neq 0$$

because of the linear independence of p and q .

We shall give here some reasons for undertaking the present investigation. At that R stands for the field of the real numbers and V for the three-dimensional vector space over R (real three-dimensional vector space).

In some previous articles we have made an attempt to emphasize by concrete examples on the important role which the application of the reciprocal repers plays in various questions of the linear algebra, linear analytical geometry and analytical mechanics since this role in our opinion is usually underestimated [1] — [12]. The advantage the reciprocal vectors propose is

a double one. On the one hand, they enable us to obtain elegant solutions of a series of basic problems whose treatment without their aid is a very clumsy one. On the other hand, the Gibbs vectors give us the opportunity to build up an entirely new technics for the formulation of some situations concerning the logical foundations of the analytical mechanics. At last, being generalized for real and complex Euclidean spaces, the reciprocal repers enable us to establish far reaching geometric-mechanical analogies.

As most instructive examples for the technical advantages of the Gibbs vectors in the case of the real three-dimensional vector space we shall point out the following ones.

Let the vectors

$$(19) \quad a_v \in V \quad (v=1, 2, 3)$$

with

$$(20) \quad a_1 \times a_2 \cdot a_3 \neq 0$$

be given and

$$(21) \quad \alpha_v \in R \quad (v=1, 2, 3).$$

Then the system of vector equations

$$(22) \quad r a_v = \alpha_v, \quad (v=1, 2, 3)$$

has exactly one solution $r \in V$, namely

$$(23) \quad r = \sum_{v=1}^n \alpha_v a_v^{-1}.$$

The elegant form (23) of the Kramer formulae for the linear system of equations (22) is retained in the multidimensional case too; let H_C be a Hermite space over the field C of the complex numbers; if

$$(24) \quad \alpha_v \in C \quad (v=1, 2, \dots, n)$$

and

$$(25) \quad a_v \in H_C \quad (v=1, 2, \dots, n)$$

are linearly independent, the system of vector equations

$$(26) \quad r a_v = \alpha_v, \quad (v=1, 2, \dots, n),$$

has exactly one solution r in the n -dimensional linear subspace $H_C(a_v)_{v=1}^n$ of H_C , generated by the vectors (25), namely

$$(27) \quad r = \sum_{v=1}^n \alpha_v a_v^{-1},$$

the reciprocal reper

$$(28) \quad a_v^{-1} \in H_C \quad (v=1, 2, \dots, n)$$

of the reper (25) being appropriately defined.

Let the vectors (19) with (20) be given and

$$(29) \quad b_v \in V \quad (v=1, 2, 3).$$

Then a necessary condition for the consistency of the system of vector equations

$$(30) \quad r \times a_v = b_v \quad (v=1, 2, 3)$$

is

$$(31) \quad a_\mu b_v + a_v b_\mu = 0 \quad (\mu, v=1, 2, 3).$$

Provided (31), the system (30) with (20) has exactly one solution $r \in V$ namely

$$(32) \quad r = \frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times b_v.$$

Applied to the system of vector equations

$$(33) \quad r \times a_v = b_v \quad (v=1, 2)$$

with

$$(34) \quad a_1 \times a_2 \neq 0$$

this theorem leads to the following result. A necessary condition for the consistency of the system (33) is

$$(35) \quad a_\mu b_v + a_v b_\mu = 0 \quad (\mu, v=1, 2).$$

Provided (35), the system (33) with (34) has exactly one solution namely (32) with

$$(36) \quad a_3 = a_1 \times a_2$$

and

$$(37) \quad b_3 = a_1 \times b_2 + b_1 \times a_2.$$

The geometrical interpretation of the systems of vector equations (22) with (19) — (21); (30) with (19), (20), (29), (31); (33) with (19), (20), (29), (34), (35) is obvious. The technical advantage of the solution (23) of (22) with (19) — (21) for example over the solution

$$(38) \quad r = \frac{1}{(a_1 \times a_2 \cdot a_3)^2} \begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ a_1^2 & a_1 a_2 & a_1 a_3 & a_1 \\ a_2 a_1 & a_2^2 & a_2 a_3 & a_2 \\ a_3 a_1 & a_3 a_2 & a_3^2 & a_3 \end{vmatrix}$$

of the same system (22) of vector equations without the aid of the Gibbs vectors is also obvious.

If the vector functions

$$(39) \quad a_v = a_v(t) \in V \quad (v=1, 2, 3)$$

with (20) represent a rigid reper (a coordinate system fixed in a rigid body), i. e. iff

$$(40) \quad \frac{d}{dt} (a_\mu a_v) = 0 \quad (\mu, v=1, 2, 3),$$

there exists exactly one vector function

$$(41) \quad \bar{\omega} = \bar{\omega}(t) \in V$$

(instantaneous angular velocity of the rigid body) with the following property: for every point

$$(42) \quad \bar{p} = \bar{p}(t) \in V$$

of the body, i. e. for every vector function (42) with

$$(43) \quad \frac{d}{dt} (\bar{p} \alpha_v^{-1}) = 0 \quad (v=1, 2, 3),$$

the equality

$$(44) \quad \frac{d\bar{p}}{dt} = \bar{\omega} \times \bar{p}$$

holds. The vector (41) can be found as the solution of the system of vector equations

$$(45) \quad \frac{da_v}{dt} = \bar{\omega} \times a_v, \quad (v=1, 2, 3),$$

if (39) with (20) are given and hence

$$(46) \quad \frac{da_v}{dt} \in V \quad (v=1, 2, 3)$$

are known; the system (45) is obtained from (44) in the special case $\bar{p} = a_v$, ($v=1, 2, 3$). Because of (40) the conditions (31) are satisfied and (32) implies

$$(47) \quad \bar{\omega} = \frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times \frac{da_v}{dt}.$$

Now (43), (45) imply (44) by differentiating the identity

$$(48) \quad \bar{p} = \sum_{v=1}^3 (\bar{p} a_v^{-1}) a_v$$

with respect to t . The result (47) is a vector-analytical version of the well-known Euler theorem. Since

$$(49) \quad a_v^{-1} = a_v, \quad (v=1, 2, 3),$$

iff

$$(50) \quad a_\mu a_\nu = \begin{cases} 1 & (\mu=v) \\ 0 & (\mu \neq v) \end{cases} \quad (\mu, \nu=1, 2, 3),$$

in case of an orthonormal coordinate system (39) the instantaneous angular velocity (47) takes the form

$$(51) \quad \bar{\omega} = \frac{1}{2} \sum_{v=1}^3 a_v \times \frac{da_v}{dt},$$

or, adopting the traditional notations for the unit vectors of an invariably connected with the rigid body coordinate system, the form

$$(52) \quad \bar{\omega} = \frac{1}{2} (\bar{\xi}^0 \times \dot{\bar{\xi}}^0 + \bar{\eta}^0 \times \dot{\bar{\eta}}^0 + \bar{\zeta}^0 \times \dot{\bar{\zeta}}^0).$$

In the case (50) of an orthonormal coordinate system (39) the equality (51), because of the trivial identity

$$(53) \quad \sum_{v=1}^3 \frac{da_v}{dt} \times \frac{da_v}{dt} = 0,$$

implies

$$(54) \quad \frac{d\bar{\omega}}{dt} = \frac{1}{2} \sum_{v=1}^3 a_v \times \frac{d^2 a_v}{dt^2}.$$

It turns out now that in the general case of a coordinate system (39) which is not necessarily orthonormal the analogous of (54) equality

$$(55) \quad \bar{\epsilon} = \frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times \frac{d^2 a_v}{dt^2}$$

or the instantaneous angular acceleration

$$(56) \quad \bar{\epsilon} = \frac{d\bar{\omega}}{dt}$$

of the body still holds although the terms of the sum

$$(57) \quad \sum_{v=1}^3 \frac{da_v^{-1}}{dt} \times \frac{da_v}{dt}$$

do not necessarily vanish, contrary to the case (53): the sum (57) is also zero:

$$(58) \quad \sum_{v=1}^3 \frac{da_v^{-1}}{dt} \times \frac{da_v}{dt} = 0$$

and (47), (56) imply (55). The identity (58) is however not an absolute but a conditional one: it is true only when (40) holds. More generally, provided (39), (40), (20), the equalities

$$(59) \quad \sum_{v=1}^3 \frac{d^m a_v}{dt^m} \times \frac{d^n a_v^{-1}}{dt^n} = \sum_{v=1}^n \frac{d^m a_v^{-1}}{dt^m} \times \frac{d^n a_v}{dt^n}$$

($m, n = 0, 1, 2, \dots$) hold where

$$(60) \quad \frac{d^0 a_v}{dt^0} = a_v, \quad \frac{d^0 a_v^{-1}}{dt^0} = a_v^{-1} \quad (v = 1, 2, 3);$$

(59) with $m=n$ imply

$$(61) \quad \sum_{v=1}^3 \frac{d^n a_v}{dt^n} \times \frac{d^n a_v^{-1}}{dt^n} = 0 \quad (n=0, 1, 2, \dots),$$

i. e. (58) is a special case of (61) ($n=1$). Because of (59) the vectors (47) and (55) can also be written in the equivalent form

$$(62) \quad \bar{\omega} = \frac{1}{2} \sum_{v=1}^3 a_v \times \frac{da_v^{-1}}{dt}$$

and

$$(63) \quad \bar{\epsilon} = \frac{1}{2} \sum_{v=1}^3 a_v \times \frac{d^2 a_v^{-1}}{dt^2}$$

respectively. In the special case (50) of an orthonormal coordinate system (39) the equality (63) takes the form

$$(64) \quad \bar{\epsilon} = \frac{1}{2} \sum_{v=1}^3 a_v \times \frac{d^2 a_v}{dt^2}$$

because of (49), respectively

$$(65) \quad \bar{\epsilon} = \frac{1}{2} (\bar{\xi}^0 \times \bar{\xi}^0 + \bar{\eta}^0 \times \bar{\eta}^0 + \bar{\zeta}^0 \times \bar{\zeta}^0)$$

according to (52).

In the case

$$(66) \quad \sum_{\mu=1}^3 \sum_{v=1}^3 \left(\frac{d}{dt} (a_\mu a_v) \right)^2 \neq 0$$

one at least of the equalities (45) fails for an arbitrary function (41). Quite naturally the question arises about the existence of functions

$$(67) \quad c_v = c_v(t) \in V \quad (v=1, 2, 3)$$

and a function (44) with

$$(68) \quad \frac{da_v}{dt} = \bar{\omega} \times a_v + c_v, \quad (v=1, 2, 3),$$

i. e. with

$$(69) \quad \bar{\omega} \times a_v = \frac{da_v}{dt} - c_v \quad (v=1, 2, 3).$$

The system of equations (69) is of the form (30). A necessary condition for its consistency is

$$(70) \quad a_\mu \left(\frac{da_v}{dt} - c_v \right) + a_v \left(\frac{da_\mu}{dt} - c_\mu \right) = 0 \quad (\mu, v=1, 2, 3)$$

according to (31), i. e.

$$(71) \quad a_\mu c_\nu + a_\nu c_\mu = \frac{d}{dt} (a_\mu a_\nu) \quad (\mu, \nu = 1, 2, 3).$$

Provided (71), the system (68) has exactly one solution (41), namely

$$(72) \quad \bar{\omega} = \frac{1}{2} \sum_{\nu=1}^3 a_\nu^{-1} \times \frac{da_\nu}{dt} + \frac{1}{2} \sum_{\nu=1}^3 c_\nu + a_\nu^{-1},$$

according to (32).

From (70) it is seen that the number of the necessary and sufficient conditions for the existence of functions (67) with (68) is 6. There are hence 3 degrees of freedom for the choice of the functions (67). One possible additional defining of these quantities could be made with the aid of the requirement the vector (41) with (68) to be of the same structure in the case (66) as in the case (45). Now from (47) and (72) it follows that such a coincidence will take place exactly when

$$(73) \quad \sum_{\nu=1}^3 c_\nu \times a_\nu^{-1} = 0.$$

From the definition of the Gibbs vectors it follows that (73) is equivalent to

$$(74) \quad \sum_{\nu=1}^3 c_\nu \times (a_{\nu+1} \times a_{\nu+2}) = 0,$$

i. e. to

$$(75) \quad (a_3 c_2 - a_2 c_3) a_1 + (a_1 c_3 - a_3 c_1) a_2 + (a_2 c_1 - a_1 c_2) a_3 = 0.$$

Because of (20) the equality (75) is equivalent to

$$(76) \quad a_\mu c_\nu = a_\nu c_\mu \quad (\mu, \nu = 1, 2, 3).$$

It follows from (76) and (71) that

$$(77) \quad a_\mu c_\nu = \frac{1}{2} \frac{d}{dt} (a_\mu a_\nu) \quad (\mu, \nu = 1, 2, 3).$$

Now (77), (20) imply

$$(78) \quad c_\nu = \frac{1}{2} \sum_{\mu=1}^3 \frac{d}{dt} (a_\mu a_\nu) a_\mu^{-1} \quad (\nu = 1, 2, 3)$$

and (68), (78) imply

$$(79) \quad \frac{da}{dt} = \bar{\omega} \times a_\nu + \frac{1}{2} \sum_{\mu=1}^3 \frac{d}{dt} (a_\mu a_\nu) a_\mu^{-1}$$

($v=1, 2, 3$). Since

$$(80) \quad \sum_{\mu=1}^3 \frac{d}{dt} (a_\mu a_v) a_\mu^{-1} = \sum_{\mu=1}^3 \left(a_\mu \frac{da_v}{dt} \right) a_\mu^{-1}$$

$$+ \sum_{\mu=1}^3 \left(a_v \frac{da_\mu}{dt} \right) a_\mu^{-1} := \frac{da_v}{dt} + \sum_{\mu=1}^3 \left(a_v \frac{da_\mu}{dt} \right) a_\mu^{-1}$$

($v=1, 2, 3$) (79) implies

$$(81) \quad \frac{da_v}{dt} = 2\bar{\omega} \times a_v + \sum_{\mu=1}^3 \left(a_v \frac{da_\mu}{dt} \right) a_\mu^{-1} \quad (v=1, 2, 3).$$

When these considerations are applied to the Gibbs vectors a_v^{-1} ($v=1, 2, 3$) instead of the initial reper (39), the equalities

$$(82) \quad (a_v^{-1})^{-1} = a_v, \quad (v=1, 2, 3)$$

together with the second form (62) of (47) imply that the equalities

$$(83) \quad \frac{da_v^{-1}}{dt} = \bar{\omega} \times a_v + \frac{1}{2} \sum_{\mu=1}^3 \frac{d}{dt} (a_\mu^{-1} a_v^{-1}) a_\mu$$

($v=1, 2, 3$), analogous to (79), also hold. The equality (81) takes now the form

$$(84) \quad \frac{da_v^{-1}}{dt} = 2\bar{\omega} \times a_v^{-1} + \sum_{\mu=1}^3 \left(a_v^{-1} \frac{da_\mu^{-1}}{dt} \right) a_\mu \quad (v=1, 2, 3).$$

If (42) is a point invariably connected with the affine coordinate system (39), i. e. if the conditions (43) are satisfied, from (48) and (79) we have

$$(85) \quad \frac{d\bar{\rho}}{dt} = \sum_{v=1}^3 (\bar{\rho} a_v^{-1}) \frac{da_v}{dt} = \sum_{v=1}^3 (\bar{\rho} a_v^{-1}) \bar{\omega} \times a_v$$

$$+ \frac{1}{2} \sum_{v=1}^3 (\bar{\rho} a_v^{-1}) \sum_{\mu=1}^3 \frac{d}{dt} (a_\mu a_v) a_\mu^{-1}.$$

Since

$$(86) \quad \sum_{v=1}^3 (\bar{\rho} a_v^{-1}) \frac{d}{dt} (a_\mu a_v)$$

$$= \frac{d}{dt} \sum_{v=1}^3 (\bar{\rho} a_v^{-1}) (\tilde{a}_\mu a_v) = \frac{d}{dt} (\bar{\rho} a_\mu) \quad (\mu=1, 2, 3)$$

according to (43), (85) implies

$$(87) \quad \frac{d\bar{\rho}}{dt} = \bar{\omega} \times \bar{\rho} + \frac{1}{2} \sum_{\mu=1}^3 \frac{d}{dt} (\bar{\rho} a_\mu) a_\mu^{-1}.$$

Calculations, which will not be given here, show that the equality (87) can be written also in the form

$$(88) \quad \frac{d\bar{\rho}}{dt} = \bar{\omega} \times \bar{\rho} + \frac{1}{2} \sum_{v=1}^3 \left(\bar{\rho} \frac{da_v}{dt} \right) a_v^{-1} + \frac{1}{2} \sum_{v=1}^3 (\bar{\rho} a_v^{-1}) \frac{da_v}{dt}.$$

The equality (88) shows that the velocity of every point (42) in variably connected with the affine coordinate system (39) can be found by vector-algebraic operations over (42) instead of differentiation and is a generalization of Euler theorem (44) for the case of an affine body.

Elegant results are also obtained when the Gibbs vectors are applied to finite displacements and finite rotations of the real three-dimensional vector space V . Let

$$(89) \quad r_v \in V \quad (v=1, 2, 3)$$

$$(90) \quad \bar{\rho}_v \in V \quad (v=1, 2, 3),$$

$$(91) \quad r_\mu r_v = \bar{\rho}_\mu \bar{\rho}_v \quad (\mu, v=1, 2, 3),$$

$$(92) \quad 0 < (r_1 \times r_2 \cdot r_3) (\bar{\rho}_1 \times \bar{\rho}_2 \cdot \bar{\rho}_3).$$

The conditions (91) express the fact that the modules of the vectors (89) are equal to the modules of their equi-index vectors (90) and the angles between every two different vectors (89) are equal to the angles between their equi-index vectors (90). The condition (92) expresses the fact that the vectors (89) and (90) form two equi-orientated repers. The equalities (91) imply

$$(93) \quad (r_1 \times r_2 \cdot r_3)^2 = (\bar{\rho}_1 \times \bar{\rho}_2 \cdot \bar{\rho}_3)^2$$

and (92), (93) imply

$$(94) \quad r_1 \times r_2 \cdot r_3 = \bar{\rho}_1 \times \bar{\rho}_2 \cdot \bar{\rho}_3.$$

Provided (89) — (92), let

$$(95) \quad \bar{\rho} = \sigma (r_v, \bar{\rho}_v; r)_{v=1}^3$$

stand for the transformation of V defined as follows. If

$$(96) \quad r = \sum_{v=1}^3 \lambda_v r_v \quad (\lambda_v \in R, v=1, 2, 3),$$

let (95) map (96) in the vector

$$(97) \quad \bar{\rho} = \sum_{v=1}^3 \lambda_v \bar{r}_v.$$

Obviously

$$(98) \quad \sigma(r_v, \bar{r}_v; r_\mu)_{v=1}^3 = \bar{\rho}_\mu \quad (\mu = 1, 2, 3).$$

Since

$$(99) \quad r = \sum_{v=1}^3 (r r_v^{-1}) r_v$$

and

$$(100) \quad \bar{\rho} = \sum_{v=1}^3 (\bar{r} \bar{r}_v^{-1}) \bar{r}_v,$$

the correspondance, realized by (95), has the property

$$(101) \quad r r_v^{-1} = \bar{\rho} \bar{r}_v^{-1} \quad (v = 1, 2, 3)$$

or every vector r and its σ -image $\bar{\rho}$. Inversely, the equalities (101) could be taken as defining the mapping (95) since (101) imply

$$(102) \quad \bar{\rho} = \sum_{v=1}^3 (r r_v^{-1}) \bar{r}_v$$

in accordance with (96), (97). The transformation (95) with (89) — (92) mapping (96) in (97) and especially (89) in (90) according to (98) is called a finite displacement of V with a fixed point 0 determined by the repers (89), (90) of V with (91), (92).

These definitions can be generalized to the case of an arbitrary Euclidean space E . Let

$$(103) \quad r_v \in E \quad (v = 1, 2, \dots, n)$$

and

$$(104) \quad \bar{r}_v \in E \quad (v = 1, 2, \dots, n)$$

be two repers in E with

$$(105) \quad r_\mu r_v = \bar{\rho}_\mu \bar{r}_v \quad (\mu, v = 1, 2, \dots, n).$$

In case E is an n -dimensional Euclidean space it must moreover be supposed, that the repers (103), (104) are equi-orientated, which means by definition that

$$(106) \quad 0 < \begin{vmatrix} r_1 & \bar{r}_1 & r_1 & \bar{r}_2 & \cdots & r_1 & \bar{r}_n \\ r_2 & \bar{r}_1 & r_2 & \bar{r}_2 & \cdots & r_2 & \bar{r}_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_n & \bar{r}_1 & r_n & \bar{r}_2 & \cdots & r_n & \bar{r}_n \end{vmatrix}.$$

Then the transformation in E

$$(107) \quad \bar{\rho} = \sigma(r_v, \bar{\rho}_v, r)_{v=1}^n,$$

mapping every vector r belonging to the n -dimensional linear space $L(r,)_v^n$ generated by the reper (103) into the vector $\bar{\rho}$ belonging to the n -dimensional linear space $L(\bar{\rho}_v)_{v=1}^n$ generated by the reper (104), according to the equations

$$(108) \quad rr_v^{-1} = \bar{\rho} \bar{\rho}_v^{-1} \quad (v=1, 2, \dots, n),$$

or, which is the same, according to the equality

$$(109) \quad \bar{\rho} = \sum_{v=1}^n (r r_v^{-1}) \bar{\rho}_v,$$

is called a finite displacement in E with a fixed point 0 determined by the repers (103), (104) in E with (105).

Another possible generalization consists in avoiding the conditions (105) and using again (108) or (109) as defining the transformation (107) conditions. This transformation shoud be called a finite affine deformation in E with a fixed point 0 determined by the repers (103), (104) in E .

Returning now to the case of the real three-dimensional vector space V , given

$$(110) \quad \omega \in R$$

and

$$(111) \quad \bar{\omega}^0 \in V$$

let

$$(112) \quad \bar{\omega} = \omega \bar{\omega}^0$$

and let

$$(113) \quad \bar{\rho} = \tau(\bar{\omega}, r)$$

be the transformation of V , defined by the equality

$$(114) \quad \bar{\rho} = (1 - \cos \omega) (r \bar{\omega}^0) \bar{\omega}^0 + \sin \omega r + \sin \omega \bar{\omega}^0 \times r$$

for every $r \in V$. The transformation (113) or (114) with (110) — (112) is called a finite rotation of V at the angle ω round the axis $\bar{\omega}^0$ through 0.

It is proved that every finite displacement (95) with (89) — (92), (98) of V with a fixed point 0 is equivalent to a finite rotation (114) with (110), (111) of V , i. e. that

$$(115) \quad \sigma(r_v, \bar{\rho}_v; r)_{v=1}^3 = \tau(\bar{\omega}, r)$$

for every $r \in V$ with appropiate (110) — (112). Especially (115), (96), (97), (114) imply

$$(116) \quad \bar{\rho}_v = (1 - \cos \omega) (r_v \bar{\omega}^0) \bar{\omega}^0 + \cos \omega r_v + \sin \omega \bar{\omega}^0 \times r_v, \quad (v=1, 2, 3).$$

This theorem formalizes the well-known Euler theorem (Novi Comment. Petrop., vol. 20 (1776), § 25, p. 189) in terms of vector algebra. The Gibbs vectors enable us to give an elegant solution of the system of vector equations (116), where the vectors (89), (90) with (91), (92) are given, whereas the real number (110) and the unit vector (111) are unknown. Despite the fact, that the system of equations (116) appears to be a very complicated one, its solution, obtained by means of reciprocal repers, is a very simple one:

$$(117) \quad \sin \omega \bar{\omega}^0 = \frac{1}{2} \sum_{v=1}^3 \bar{r}_v^{-1} \times \bar{\rho}_v$$

and

$$(118) \quad \cos \omega = \frac{1}{2} \left(\sum_{v=1}^3 \bar{r}_v^{-1} \bar{\rho}_v - 1 \right).$$

The equalities (117), (118) are a vector version of the well-known Rodrigues formula, first given in [13]. Of course an intermediate step in the proof consists in establishing the conditional, i. e. provided (91), (92), identity

$$(119) \quad \left(\sum_{v=1}^3 \bar{r}_v^{-1} \times \bar{\rho}_v \right)^2 + \left(\sum_{v=1}^3 \bar{r}_v^{-1} \bar{\rho}_v - 1 \right)^2 = 4.$$

The relation between finite displacements and finite rotations of the real three-dimensional vector space V could be also established in the following manner. Let

$$(120) \quad r_v \in V \quad (v=1, 2),$$

$$(121) \quad \bar{\rho}_v \in V \quad (v=1, 2),$$

$$(122) \quad r_\mu r_v = \bar{\rho}_\mu \bar{\rho}_v \quad (\mu, v=1, 2),$$

$$(123) \quad r_1 \times r_2 \neq 0.$$

(120) — (122) imply

$$(124) \quad (r_1 \times r_2)^2 = (\bar{\rho}_1 \times \bar{\rho}_2)^2.$$

(123), (124) imply

$$(125) \quad \bar{\rho}_1 \times \bar{\rho}_2 \neq 0.$$

Now let

$$(126) \quad r_3 = r_1 \times r_2, \quad \bar{\rho}_3 = \bar{\rho}_1 \times \bar{\rho}_2.$$

Then

$$(127) \quad r_v r_3 = 0, \quad \bar{\rho}_v \bar{\rho}_3 = 0 \quad (v=1, 2),$$

$$(128) \quad r_3^2 = (r_1 \times r_2)^2, \quad \bar{\rho}_3^2 = (\bar{\rho}_1 \times \bar{\rho}_2)^2.$$

(120), (121), (126) imply (89), (90); (122), (127), (128), (124) imply (92). Then the result for finite displacements of V could be reformulated for the

vectors (120), (121) with (122), (123) instead of the vectors (89), (90) with (91), (92).

The Gibbs vectors play an important role also in other problems of the analytical mechanics, for example in the three- and multidimensional point kinematics, in the axiomatical building up of affine and rigid body kinematics in real and complex Hilbert spaces, in the axiomatical approach to the real three-dimensional linear analytic geometry etc. Now it turns out that all these considerations work mutatis mutandis in more general situations. One of the aims of the present article is to prepare the ground for such investigations, in the first place for developing an algebraic theory of arrows, an axiomatic approach to Pythagorean analytic geometries, for defining finite rotations in discontinuous vector spaces etc.

Let

$$\begin{array}{ll} (129) & \mathbf{a}_v \in V_F \\ (130) & \mathbf{b}_v \in V_F \\ (131) & \mathbf{a}_v \in V_F \\ (132) & \mathbf{b}_v \in V_F \end{array} \quad (v=1, 2), \quad (v=1, 2, 3), \quad (v=1, 2, 3).$$

Then:

Pr 1. (129), (130) imply

$$(133) \quad \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{b}_1 \times \mathbf{b}_2 = (\mathbf{a}_1 \mathbf{b}_1) (\mathbf{a}_2 \mathbf{b}_2) - (\mathbf{a}_1 \mathbf{b}_2) (\mathbf{a}_2 \mathbf{b}_1).$$

$$\begin{aligned} \text{Dm. } & \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{b}_1 \times \mathbf{b}_2 = (\mathbf{a}_1 \times \mathbf{a}_2) \times \mathbf{b}_1 \cdot \mathbf{b}_2 = ((\mathbf{a}_1 \mathbf{b}_1) \mathbf{a}_2 - (\mathbf{a}_2 \mathbf{b}_1) \mathbf{a}_1) \mathbf{b}_2 \\ & = (\mathbf{a}_1 \mathbf{b}_1) (\mathbf{a}_2 \mathbf{b}_2) - (\mathbf{a}_1 \mathbf{b}_2) (\mathbf{a}_2 \mathbf{b}_1). \end{aligned}$$

Pr 2. (129), (130) imply

$$(134) \quad \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \mathbf{a}_1 \mathbf{b}_1 & \mathbf{a}_1 \mathbf{b}_2 \\ \mathbf{a}_2 \mathbf{b}_1 & \mathbf{a}_2 \mathbf{b}_2 \end{vmatrix}.$$

Dm. Pr 1

Pr 3. (129) imply

$$(135) \quad (\mathbf{a}_1 \times \mathbf{a}_2)^2 = \mathbf{a}_1^2 \mathbf{a}_2^2 - (\mathbf{a}_1 \mathbf{a}_2)^2.$$

Dm. Pr 1.

Pr 4. (129) imply

$$(136) \quad (\mathbf{a}_1 \times \mathbf{a}_2)^2 = \begin{vmatrix} \mathbf{a}_1^2 & \mathbf{a}_1 \mathbf{a}_2 \\ \mathbf{a}_2 \mathbf{a}_1 & \mathbf{a}_2^2 \end{vmatrix}.$$

Dm. Pr 3.

Pr 5. (129),

$$(137) \quad \mathbf{a}_3 \text{ sgn: } \mathbf{a}_1 \times \mathbf{a}_2$$

imply

$$(138) \quad \mathbf{a}_v \mathbf{a}_3 = 0 \quad (v=1, 2).$$

Dm. Clear.

Pr 6. (129), (137) imply

$$(139) \quad \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 = (\mathbf{a}_1 \times \mathbf{a}_2)^2.$$

Dm. Clear.

Pr 7. (131),

$$(140) \quad r \in V_F$$

imply

$$(141) \quad (a_1 \times a_2 \cdot a_3) r = (a_2 \times a_3 \cdot r) a_1 + (a_3 \times a_1 \cdot r) a_2 + (a_1 \times a_2 \cdot r) a_3.$$

Dm. (141) follows from

$$(142) \quad (a_1 \times a_2) \times (a_3 \times r) = (a_1 \cdot a_3 \times r) a_2 - (a_2 \cdot a_3 \times r) a_1$$

and

$$(143) \quad (a_1 \times a_2) \times (a_3 \times r) = (a_1 \times a_2 \cdot r) a_3 - (a_1 \times a_2 \cdot a_3) r.$$

Pr 8. (131), (132) imply

$$(144) \quad (a_1 \times a_2 \cdot a_3) (b_1 \times b_2 \cdot b_3) = \begin{vmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{vmatrix}.$$

Dm. Pr 7, Pr 1 imply

$$(145) \quad (a_1 \times a_2 \cdot a_3) b_1 \times b_2 \\ = \begin{vmatrix} a_2 b_1 & a_2 b_2 \\ a_3 b_1 & a_3 b_2 \end{vmatrix} a_1 + \begin{vmatrix} a_3 b_1 & a_3 b_2 \\ a_1 b_1 & a_1 b_2 \end{vmatrix} a_2 + \begin{vmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{vmatrix} a_3,$$

whence (144).

Pr 9. (131) imply

$$(146) \quad (a_1 \times a_2 \cdot a_3)^2 = \begin{vmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{vmatrix}.$$

Dm. Pr 8.

Pr 10. (131), (132),

$$(147) \quad a_{r+3} \text{ sgn: } a_r \quad (\nu = 1, 2),$$

and

$$(148) \quad b_{r+3} \text{ sgn: } b_r \quad (\nu = 1, 2)$$

imply

$$(149) \quad \sum_{r=1}^3 (a_{r+1} \times a_{r+2}) \times b_r = \sum_{r=1}^3 (a_{r+2} b_{r+1} - a_{r+1} b_{r+2}) a_r.$$

Dm. (149) follows from

$$(150) \quad \sum_{r=1}^3 (a_{r+1} \times a_{r+2}) \times b_r = \sum_{r=1}^3 (a_{r+1} b_r) a_{r+2}$$

$$-\sum_{r=1}^3 (a_{r+2} b_r) a_{r+1} = \sum_{r=1}^3 (a_{r+2} b_{r+1}) a_r - \sum_{r=1}^3 (a_{r+1} b_{r+2}) a_r.$$

Pr 11. (131), (147) imply

$$(151) \quad \sum_{r=1}^3 a_r \times (a_{r+1} \times a_{r+2}) = 0.$$

Dm. Pr 10.

Df 1. The dyad (129) is called *orthogonal* iff

$$(152) \quad a_1 a_2 = 0.$$

Pr 12. (129), (152) imply

$$(153) \quad (a_1 \times a_2)^2 = a_1^2 a_2^2.$$

Dm. Pr 3.

Df 2. The triad (131) is called *orthogonal* iff

$$(154) \quad a_\mu a_\nu = 0 \quad (\mu, \nu = 1, 2, 3; \mu \neq \nu).$$

Pr 13. (131), (154) imply

$$(155) \quad (a_1 \times a_2 \cdot a_3)^2 = a_1^2 a_2^2 a_3^2.$$

Dm. Pr 9.

Df 3. The dyad (129) is called a *reper* in V_F iff

$$(156) \quad a_1 \times a_2 \neq 0.$$

Pr 14. (129), (152) imply (156) iff

$$(157) \quad a_\nu \neq 0 \quad (\nu = 1, 2).$$

Dm. Pr 12.

Df 4. The triad (131) is called a *reper* of V_F iff

$$(158) \quad a_1 \times a_2 \cdot a_3 \neq 0.$$

Pr 15. (131), (154) imply (158) iff

$$(159) \quad a_\nu \neq 0 \quad (\nu = 1, 2, 3).$$

Dm. Pr 13.

Df 5. A reper (131) of V_F is called *right orientated* iff

$$(160) \quad 0 < a_1 \times a_2 \cdot a_3.$$

Df 6. A reper (131) of V_F is called *left orientated* iff

$$(161) \quad a_1 \times a_2 \cdot a_3 < 0.$$

Pr 16. (129), (137), (156) imply (160).

Dm. Pr 6.

Df 7. The repers (131), (132) of V_F are called *equi-orientated* iff

$$(162) \quad 0 < (a_1 \times a_2 \cdot a_3) (b_1 \times b_2 \cdot b_3).$$

Df 8. The repers (131), (132) of V_F are called *contra-orientated* iff

$$(163) \quad (a_1 \times a_2 \cdot a_3)(b_1 \times b_2 \cdot b_3) < 0.$$

Pr 17. (129),

$$(164) \quad a_\mu a_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \quad (\mu, \nu = 1, 2)$$

imply

$$(165) \quad (a_1 \times a_2)^2 = 1.$$

Dm. Pr 12.

Pr 18. (129), (164) imply (156).

Dm. Pr 17.

Df 9. The reper (129) in V_F is called *orthonormal* iff (164).

Pr 19. (129), (137), (164) imply

$$(166) \quad a_1 \times a_2 \cdot a_3 = 1.$$

Dm. Pr 6, Pr 17.

Pr 20. (129), (137), (164) imply

$$(167) \quad a_\mu a_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \quad (\mu, \nu = 1, 2, 3)$$

Dm. Pr 5, Pr 19.

Pr 21. (131), (167) imply

$$(168) \quad (a_1 \times a_2 \cdot a_3)^2 = 1.$$

Dm. Pr 13.

Pr 22. (131), (168) imply (158).

Dm. Pr 21.

Df 10. The reper (131) of V_F is called *orthonormal* iff (167).

Sgn 1. a_ν^{-1} sgn: $\frac{a_{\nu+1} \times a_{\nu+2}}{a_1 \times a_2 \cdot a_3}$ ($\nu = 1, 2, 3$) iff (131), (147), (158).

Df. 11. a_ν^{-1} ($\nu = 1, 2, 3$) are called the *reciprocal* or *Gibbs vectors* to the vectors (131) or else the reper

$$(169) \quad a_\nu^{-1} \in V_F \quad (\nu = 1, 2, 3)$$

(s. Pr 25 below) is called the *reciprocal* or *Gibbs reper* to the reper (131).

The following propositions describe the basic properties of the reciprocal repers in V_F

Pr 23. (131), (158) imply

$$(170) \quad a_\mu a_\nu^{-1} = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \quad (\mu, \nu = 1, 2, 3).$$

Dm. Sgn 1.

Pr 24. (131), (158) imply

$$(171) \quad (a_1 \times a_2 \cdot a_3)(a_1^{-1} \times a_2^{-1} \cdot a_3^{-1}) = 1.$$

Dm. Pr 8, Pr 23.

Pr 25. (131), (158) imply

$$(172) \quad a_1^{-1} \times a_2^{-1} \cdot a_3^{-1} \neq 0.$$

Dm. Pr 24.

Pr 26. (131), (158) imply (160) iff

$$(173) \quad 0 < a_1^{-1} \times a_2^{-1} \cdot a_3^{-1}.$$

Dm. Pr 24.

Pr 27. (131), (158) imply (161) iff

$$(174) \quad a_1^{-1} \times a_2^{-1} \cdot a_3^{-1} < 0.$$

Dm. Pr 24.

Pr 28. (131), (158) imply (168) iff

$$(175) \quad a_1 \times a_2 \cdot a_3 = a_1^{-1} \times a_2^{-1} \cdot a_3^{-1}.$$

Dm. Pr 24.

Pr 29. (131), (158) imply

$$(176) \quad (a_\nu^{-1})^{-1} = a_\nu, \quad (\nu = 1, 2, 3).$$

Dm. Pr 25, Sgn 1.

Pr 30. (131), (158) imply

$$(177) \quad \sum_{\nu=1}^3 a_\nu \times a_\nu^{-1} = 0.$$

Dm. Pr 11, Sgn 1.

Pr 31. (131), (140), (158) imply

$$(178) \quad \sum_{\nu=1}^3 (ra_\nu) a_\nu^{-1} = \sum_{\nu=1}^3 (ra_\nu^{-1}) a_\nu.$$

Dm. Pr 30.

Pr 32. (131), (158),

$$(179) \quad 1 \leq \nu \leq 3$$

imply: a necessary and sufficient condition for

$$(180) \quad a_\mu \times a_\nu^{-1} = 0$$

is

$$(181) \quad a_\mu a_\nu = 0 \quad (\mu = 1, 2, 3; \mu \neq \nu).$$

Dm. Sgn 1.

Pr 33. (131), (158), (179) imply: a necessary and sufficient condition for (180) is

$$(182) \quad a_\mu^{-1} a_\nu^{-1} = 0 \quad (\mu = 1, 2, 3; \mu \neq \nu).$$

Dm. Pr 25, Pr 29, Pr 32.

Pr 34. (131), (158), (179) imply: a necessary and sufficient condition for (181) is (182).

Dm. Pr 32, Pr 33.

Pr 35. (131), (158), (179) imply: a necessary and sufficient condition for

$$(183) \quad a_v = a_v^2 a_v^{-1}$$

is (181).

Dm. Pr 32, Pr 25, Pr 23.

Pr 36. (131), (158), (179) imply: a necessary and sufficient condition for (183) is (182).

Dm. Pr 34, Pr 35.

Pr 37. (131), (158), (179) imply: a necessary and sufficient condition for (183)

$$(184) \quad a_v^{-1} = (a_v^{-1})^3 a_v$$

is (181).

Dm. Pr 32, Pr 23.

Pr 38. (131), (158), (179) imply: a necessary and sufficient condition for (184) is (182).

Dm. Pr 33, Pr 37.

Pr 39. (131), (158), (179) imply: a necessary and sufficient condition for (185)

$$a_v^2 (a_v^{-1})^3 = 1$$

is (181).

Dm. Pr 23, Pr 32.

Pr 40. (131), (158), (179) imply: a necessary and sufficient condition for (185) is (182).

Dm. Pr 34, Pr 39.

Pr 41. (131) imply: a necessary and sufficient condition for

$$(186) \quad a_v^{-1} = a_v \quad (v=1, 2, 3)$$

is (167).

Dm. Pr 22, Pr 35.

Pr 42. (131), (147), (167) imply

$$(187) \quad a_v = a_{v+1} \times a_{v+2} \quad (v=1, 2, 3)$$

iff (166).

Dm. Pr 21, Sgn 1, Pr 41.

Pr 43. (131), (147), (167) imply

$$(188) \quad a_v = a_{v+2} \times a_{v+1} \quad (v=1, 2, 3)$$

iff

$$(189) \quad a_1 \times a_2 \cdot a_3 = -1.$$

Dm. Pr 21, Sgn 1, Pr 41.

Pr 44. (129), (137), (156) imply

$$(190) \quad a_1^{-1} = \frac{a_2 \times (a_1 \times a_2)}{(a_1 \times a_2)^2},$$

$$(191) \quad a_2^{-1} = \frac{(a_1 \times a_2) \times a_1}{(a_1 \times a_2)^2},$$

$$(192) \quad \mathbf{a}_3^{-1} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{(\mathbf{a}_1 \times \mathbf{a}_2)^2}.$$

Dm. Pr 6, Sgn 1.

Pr 45. (129), (137), (156) imply

$$(193) \quad \mathbf{a}_3^{-1} = \mathbf{a}_1^{-1} \times \mathbf{a}_2^{-1}.$$

Dm. Pr 44.

Pr 46. (129), (137), (156) imply

$$(194) \quad \mathbf{a}_1^{-1} \mathbf{a}_2^{-1} = -\frac{\mathbf{a}_1 \mathbf{a}_2}{(\mathbf{a}_1 \times \mathbf{a}_2)^2}.$$

Dm. Pr 44.

Pr 47. (129), (137), (156) imply

$$(195) \quad \mathbf{a}_v^{-1} \mathbf{a}_3^{-1} = 0 \quad (v=1, 2).$$

Dm. Pr 44.

Pr 48. (129), (137), (156) imply

$$(196) \quad \mathbf{a}_3 \times \mathbf{a}_3^{-1} = 0.$$

Dm. Pr 44.

Pr 49. (129), (137), (156) imply

$$(197) \quad \mathbf{a}_1^{-1} \times \mathbf{a}_2^{-1} \cdot \mathbf{a}_3^{-1} = (\mathbf{a}_1^{-1} \times \mathbf{a}_2^{-1})^2.$$

Dm. Pr 45.

Pr 50. (129), (137), (156) imply

$$(198) \quad (\mathbf{a}_1 \times \mathbf{a}_2)^2 (\mathbf{a}_1^{-1} \times \mathbf{a}_2^{-1})^2 = 1.$$

Dm. Pr 6, Pr 49, Pr 24.

Pr 51. (129), (137), (156) imply: a necessary and sufficient condition for

$$(199) \quad \mathbf{a}_v \times \mathbf{a}_2^{-1} = 0 \quad (v=1, 2)$$

is (152).

Dm. Pr 6, Pr 5, Pr 32.

Pr 52. (129), (137), (156) imply: a necessary and sufficient condition for (199) is

$$(200) \quad \mathbf{a}_1^{-1} \mathbf{a}_2^{-1} = 0.$$

Dm. Pr 46, Pr 51.

Pr 53. (129), (137) imply: a necessary and sufficient condition for (186) is (164).

Dm. Pr 20, Pr 5, Pr 48, Pr 51, Pr 35.

Pr 54. (131), (140), (158) imply

$$(201) \quad r = \sum_{v=1}^3 (r \mathbf{a}_v^{-1}) \mathbf{a}_v.$$

Dm. There exist

$$(202) \quad \alpha_v \in F \quad (v=1, 2, 3)$$

with

$$(203) \quad r = \sum_{v=1}^3 \alpha_v a_v.$$

Now Pr 23 implies

$$(204) \quad \alpha_v = r a_v^{-1} \quad (v=1, 2, 3)$$

Pr 55. (131), (140), (158) imply

$$(205) \quad r = \sum_{v=1}^3 (r a_v) a_v^{-1}.$$

Dm. Pr 54, Pr 31.

Pr 56. (131), (140), (167) imply

$$(206) \quad r = \sum_{v=1}^3 (r a_v) a_v.$$

Dm. Pr 21, Pr 41, Pr 54.

Pr 57. (131), (158),

$$(207) \quad p \in V_F,$$

$$(208) \quad q \in V_F$$

imply

$$(209) \quad pq = \sum_{v=1}^3 (p a_v) (q a_v^{-1}).$$

Dm. Pr 54, Pr 55, Pr 23.

Pr 58. (131), (167), (207), (208) imply

$$(210) \quad pq = \sum_{v=1}^3 (p a_v) (q a_v).$$

Dm. Pr 41, Pr 57.

Pr 59. (131), (158), (207), (208) imply

$$(211) \quad p \times q = \frac{1}{a_1 \times a_2 \cdot a_3} \begin{vmatrix} a_1 & a_2 & a_3 \\ pa_1 & pa_2 & pa_3 \\ qa_1 & qa_2 & qa_3 \end{vmatrix}.$$

Dm. Pr 54, Sgn 1 imply

$$\begin{aligned}
 (212) \quad p \times q &= \sum_{r=1}^3 (p \times q \cdot a_r^{-1}) a_r \\
 &= \frac{1}{a_1 \times a_2 \cdot a_3} \sum_{r=1}^3 (p \times q \cdot a_{r+1} \times a_{r+2}) a_r \\
 &= \frac{1}{a_1 \times a_2 \cdot a_3} \sum_{r=1}^3 ((pa_{r+1})(qa_{r+2}) - (pa_{r+2})(qa_{r+1})) a_r.
 \end{aligned}$$

whence (211) according to (147).

Pr 60. (131), (158), (207), (208) imply

$$(213) \quad p \times q = (a_1 \times a_2 \cdot a_3) \begin{vmatrix} a_1^{-1} & a_2^{-1} & a_3^{-1} \\ pa_1^{-1} & pa_2^{-1} & pa_3^{-1} \\ qa_1^{-1} & qa_2^{-1} & qa_3^{-1} \end{vmatrix}.$$

Dm. Pr 24, Pr 59.

Pr 61. (131), (132), (147), (148), (158) imply

$$(214) \quad \sum_{r=1}^3 a_r^{-1} \times b_r = \frac{1}{a_1 \times a_2 \cdot a_3} \sum_{r=1}^3 (a_{r+2} b_{r+1} - a_{r+1} b_{r+2}) a_r.$$

Dm. Pr 10, Sgn 1.

Pr 62. (131), (132), (158) imply

$$(215) \quad \sum_{r=1}^3 a_r^{-1} \times b_r = 0$$

iff

$$(216) \quad a_\mu b_\nu = a_\nu b_\mu \quad (\mu, \nu = 1, 2, 3; \mu \neq \nu).$$

Dm. Pr 61.

Pr 63. (131), (132), (158) imply (215) iff

$$(217) \quad b_r = \sum_{\mu=1}^3 (a_\nu b_\mu) a_\mu^{-1} \quad (\nu = 1, 2, 3).$$

Dm. Pr 55 implies

$$(218) \quad b_r = \sum_{\mu=1}^3 (a_\mu b_\nu) a_\mu^{-1} \quad (\nu = 1, 2, 3).$$

Now Pr 62.

Pr 64. (129), (130), (137), (156),

$$(219) \quad b_3 = a_1 \times b_2 + b_1 \times a_2$$

imply (215) iff

$$(220) \quad a_1 \cdot b_2 = a_2 \cdot b_1,$$

$$(221) \quad a_1 \times a_2 \cdot b_v = 0 \quad (v=1, 2).$$

Dm. Pr 62.

Pr 65. (131), (158), (202) imply: the system of vector equations

$$(222) \quad r a_v = \alpha_v \quad (v=1, 2, 3)$$

has exactly one solution (140), namely

$$(223) \quad r = \sum_{v=1}^3 \alpha_v a_v^{-1}.$$

Dm. Pr 55, Pr 23.

Pr 66. (131), (158) imply: the system of vector equations

$$(224) \quad r a_v = 0 \quad (v=1, 2, 3)$$

has exactly one solution (140), namely

$$(225) \quad r = 0.$$

Dm. Pr 65.

Pr 67. (131), (167), (202) imply: the system of vector equations (222) has exactly one solution (140), namely (203).

Dm. Pr 41, Pr 65.

Pr 68. (131), (132), (158) imply

$$(226) \quad a_\mu \cdot b_v = \begin{cases} 1 & (\mu=v) \\ 0 & (\mu \neq v) \end{cases} \quad (\mu, v=1, 2, 3)$$

iff

$$(227) \quad b_v = a_v^{-1} \quad (v=1, 2, 3).$$

Dm. Pr 23, Pr 65.

Pr 69. (129), (156),

$$(228) \quad \alpha_v \in F \quad (v=1, 2)$$

imply: the system of vector equations

$$(229) \quad r a_v = \alpha_v \quad (v=1, 2)$$

$$(230) \quad r \cdot a_1 \times a_2 = 0$$

has exactly one solution (140), namely

$$(231) \quad r = \frac{a_1 \times a_2}{(a_1 \times a_2)^2} \times (\alpha_2 a_1 - \alpha_1 a_2).$$

Dm. Pr 6, Pr 44, Pr 65.

Pr 70. (129), (156) imply : the system of vector equations

$$(232) \quad r a_v = 0 \quad (v=1, 2),$$

(230) has exactly one solution (140), namely (225).

Dm. Pr 69.

Pr 71. (129), (164), (228) imply : the system of vector equations (229),

(230) has exactly one solution (140), namely

$$(233) \quad r = a_1 a_1 + a_2 a_2.$$

Dm. Pr 53, Pr 67.

Pr 72. (129), (130), (140) imply : a necessary condition for the consistency of the system of vector equations

$$(234) \quad r \times a_v = b_v \quad (v=1, 2)$$

is

$$(235) \quad a_\mu b_v + a_v b_\mu = 0 \quad (\mu, v=1, 2).$$

Dm. (234) imply

$$(236) \quad \begin{aligned} a_\mu b_v + a_v b_\mu &= r \times a_v \cdot a_\mu + r \times a_\mu \cdot a_v \\ &= r \cdot a_v \times a_\mu + r \cdot a_\mu \times a_v = r (a_v \times a_\mu + a_\mu \times a_v) = 0 \quad (\mu, v=1, 2). \end{aligned}$$

Pr 73. (131), (132), (140) imply : a necessary condition for the consistency of the system of vector equations

$$(237) \quad r \times a_v = b_v \quad (v=1, 2, 3)$$

is

$$(238) \quad a_\mu b_v + a_v b_\mu = 0 \quad (\mu, v=1, 2, 3).$$

Dm. Pr 72.

Pr 74. (131), (132), (140) imply : a necessary condition for the consistency of the system of vector equations (237) is

$$(239) \quad b_1 \times b_2 \cdot b_3 = 0.$$

Dm. (237) imply

$$(240) \quad \begin{aligned} b_1 \times b_2 \cdot b_3 &= (r \times a_1) \times (r \times a_2) \cdot (r \times a_3) \\ &= (r \times a_1 \cdot a_2) (r \cdot r \times a_3) = 0. \end{aligned}$$

Pr 75. (131), (132), (140), (147) imply : a necessary condition for the consistency of the system of vector equations (237) is

$$(241) \quad \sum_{v=1}^3 a_{v+1} \times a_{v+2} \cdot b_v = 0.$$

Dm. (237), Pr 11 imply

$$(242) \quad \sum_{v=1}^3 a_{v+1} \times a_{v+2} \cdot b_v = \sum_{v=1}^3 a_{v+1} \times a_{v+2} \cdot r \times a_v$$

$$= \sum_{v=1}^3 r \cdot a_v \times (a_{v+1} \times a_{v+2}) = r \sum_{v=1}^3 a_v \times (a_{v+1} \times a_{v+2}) = 0.$$

Pr 76. (131), (132), (140), (158) imply: a necessary condition for the consistency of the system of vector equations (237) is

$$(243) \quad \sum_{v=1}^3 a_v^{-1} \cdot b_v = 0.$$

Dm. Pr 75, Sgn 1.

Pr 77. (129), (130), (156) imply: the system of vector equations (234) does not admit more than one solution (140).

Dm. Let

$$(244) \quad r_v \in V_r \quad (v=1, 2),$$

$$(245) \quad r_\mu \times a_v = b_v \quad (\mu, v=1, 2),$$

$$(246) \quad \bar{\rho} = r_1 - r_2.$$

Then

$$(247) \quad \bar{\rho} \times a_v = 0 \quad (v=1, 2),$$

whence

$$(248) \quad \bar{\rho} = \alpha_1 a_1 + \alpha_2 a_2 \quad (v=1, 2)$$

with (228) since (157) according to (156); (248) imply

$$(249) \quad 0 = \bar{\rho} \times \bar{\rho} = (\alpha_1 \alpha_2) a_1 \times a_2;$$

(156), (249) imply

$$(250) \quad \alpha_1 \alpha_2 = 0,$$

whence

$$(251) \quad \bar{\rho} = 0$$

according to (248), i. e.

$$(252) \quad r_1 = r_2$$

according to (246).

Pr 78. (131), (132), (158), (238) imply: the system of vector equations (237) has exactly one solution (140), namely

$$(253) \quad r = \frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times b_v.$$

Dm. Pr 77 and Pr 23 implies

$$(254) \quad \left(\sum_{v=1}^3 a_v^{-1} \times b_v \right) \times a_\mu = b_\mu - \sum_{v=1}^3 (a_\mu \cdot b_v) a_v^{-1} \quad (\mu=1, 2, 3).$$

Now (25), (238) Pr 55 imply

$$(255) \quad \left(\frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times b_v \right) \times a_\mu = b_\mu \quad (\mu = 1, 2, 3).$$

Pr 7. (131), (132), (140), (158) imply: (238) is a sufficient condition for the consistency of the system of vector equations (237).

Dm. Pr 78.

Pr 8. (131), (132), (158), (238) imply: the system of vector equations (237) has exactly one solution (140), namely

$$(256) \quad r = \frac{1}{a_1 \times a_2 \cdot a_3} \sum_{v=1}^3 (a_{v+2} b_{v+1}) a_v$$

with (147)-(148).

Dm. Pr 78, Pr 61.

Pr. 8 (131), (132), (158), (238) imply: the system of vector equations (237) has exactly one solution (140), namely

$$(257) \quad r = \frac{-1}{a_1 \times a_2 \cdot a_3} \sum_{v=1}^3 (a_{v+1} b_{v+2}) a_v$$

with (147)-(148).

Dm. Pr 80.

Pr 82. (31), (132), (166), (238) imply: the system of vector equations (237) has exactly one solution (140), namely

$$(258) \quad r = \sum_{v=1}^3 (a_{v+2} b_{v+1}) a_v$$

with (147)-(148).

Dm. Pr 80.

Pr 83. (31), (132), (189), (238) imply: the system of vector equations (237) has exactly one solution (140), namely

$$(259) \quad r = \sum_{v=1}^3 (a_{v+1} b_{v+2}) a_v$$

with (147)-(148).

Dm. Pr 81.

Pr 84. (31), (158) imply: the system of vector equations

$$(260) \quad r \times a_v = 0 \quad (v = 1, 2, 3)$$

has exactly one solution (140), namely (225).

Dm. Pr 78.

Pr 85. (131), (132), (167), (238) imply: the system of vector equations (237) has exactly one solution (140), namely

$$(261) \quad r = -\frac{1}{2} \sum_{v=1}^3 a_v \times b_v.$$

Dm. Pr 41, Pr 78.

Pr 86. (129), (130), (156), (235) imply: the system of vector equations (234) has exactly one solution (140), namely (253) with (137), (219).

Dm. (137), (219) imply

$$(262) \quad a_v b_3 + a_3 b_v = 0 \quad (v=1, 2).$$

(137), (219), (235) imply

$$(263) \quad a_3 b_3 = a_1 \times a_2 \cdot (a_1 \times b_3 + b_1 \times a_2);$$

(235), (262), (263) imply (238). The statement now follows from Pr 16, Pr 61, Pr 62.

Pr 87. (129), (130), (140), (156) imply: (235) is a sufficient condition for the consistency of the system of vector equations (234).

Dm. Pr 86.

Pr 88. (129), (130), (156), (235) imply: the system of vector equations (234) has exactly one solution (140), namely

$$(264) \quad r = -\frac{1}{(a_1 \times a_2)^2} \sum_{v=1}^3 (a_{v+2} b_{v+1}) a_v$$

with (137), (219), (147), (148).

Dm. Pr 6, Pr 86, Pr 61.

Pr 89. (129), (130), (156), (235) imply: the system of vector equations (234) has exactly one solution (140), namely

$$(265) \quad r = \frac{-1}{(a_1 \times a_2)^2} \sum_{v=1}^3 (a_{v+1} b_{v+2}) a_v$$

with (137), (219), (147), (148).

Dm. Pr. 88.

Pr 90. (129), (130), (165), (235) imply: the system of vector equations (234) has exactly one solution (140), namely (258) with (137), (219), (147), (148).

Dm. Pr 88.

Pr 91. (129), (130), (165), (235) imply: the system of vector equations (234) has exactly one solution (140), namely

$$(266) \quad r = -\sum_{v=1}^3 (a_{v+1} b_{v+2}) a_v$$

with (137), (219), (147), (148).

Dm. Pr 89.

Pr. 92. (129), (156) imply: the system of vector equations

$$(267) \quad r \times a_v = 0 \quad (v = 1, 2)$$

has exactly one solution (140), namely (225).

Dm. Pr 86.

Pr 93. (129), (130), (164), (235) imply: the system of vector equations

(234) has exactly one solution (140), namely (261) with (137), (219).

Dm. Pr 53, Pr 86.

Pr 94. (129), (130), (156), (235) imply: the system of vector equations

(234) has exactly one solution (140), namely

$$(268) \quad r = \frac{1}{(a_1 \times a_2)^2} ((a_1 \times a_2 \cdot b_2) a_1 + (a_2 \times a_1 \cdot b_1) a_2 + (a_1 \cdot b_1) a_1 \times a_2).$$

Dm. Pr 86, (137), (219).

Pr 95. (127), (130), (156), (235) imply: the system of vector equations

(234) has exactly one solution (140), namely

$$(269) \quad r = a_1^{-1} \times b_1 + a_2^{-1} \times b_2 + (a_1 \cdot b_2) a_3^{-1}$$

with (137).

Dm. Pr 94, Pr 44 and

$$(270) \quad \begin{aligned} & (a_1 \times a_2 \cdot b_2) a_1 + (a_2 \times a_1 \cdot b_1) a_2 + (a_1 \cdot b_1) a_1 \times a_2 \\ &= ((a_1 \cdot b_1) a_1 \times a_2 - (a_1 \times a_2 \cdot b_1) a_2) + ((a_1 \times a_2 \cdot b_2) a_1 \\ & - (a_1 \cdot b_2) a_1 \times a_2) + (a_1 \cdot b_2) a_1 \times a_2 = (a_2 \times (a_1 \times a_2)) \times b_1 \\ & + ((a_1 \times a_2) \times a_1) \times b_2 + (a_1 \cdot b_2) a_1 \times a_2. \end{aligned}$$

Pr 96. (131), (132), (140), (156),

$$(271) \quad a_1 \times a_2 \cdot a_3 = 0$$

imply: necessary conditions for the consistency of the system of vector equations (237) are (235) and

$$(272) \quad a_3 = \alpha_1 a_1 + \alpha_2 a_2,$$

$$(273) \quad b_3 = \alpha_1 b_1 + \alpha_2 b_2$$

with (228).

Dm. The consistency of the system of vector equations (237) implies the consistency of the system of vector equations (234), whence (235) according to Pr 72. Since (271), (156) imply (272) with appropriate (228), (237) implies

$$(274) \quad \begin{aligned} b_3 &= r \times a_3 = r \times (\alpha_1 a_1 + \alpha_2 a_2) \\ &= \alpha_1 r \times a_1 + \alpha_2 r \times a_2 = \alpha_1 b_1 + \alpha_2 b_2. \end{aligned}$$

Pr 97. (131), (132), (140), (156) imply: (235), (272), (273) with (228) are sufficient conditions for the consistency of the system of vector equati-

ons (237): (235), (272), (273) imply that the system (237) has exactly one solution (140), namely (268).

Dm. Pr 94 and let (14) be a solution of the system of vector equations (234); then

$$(275) \quad r \times a_v = r \times (\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1 r \times a_1 + \alpha_2 r \times a_2 = \alpha_1 b_1 + \alpha_2 b_2$$

according to (272), (273) imply: (140) is a solution of the third vector equation (237).

Pr 98. (131), (132), (158) imply: the system

$$(276) \quad r \times a_v = b_v - \sum_{\lambda=1}^3 (a_v \cdot b_\lambda) a_\lambda^{-1} \quad (v=1, 2, 3)$$

has exactly one solution (140), namely

$$(277) \quad r = \sum_{v=1}^3 a_v^{-1} \times b_v.$$

Dm. Pr 23 implies

$$(278) \quad a_\mu \left(b_v - \sum_{\lambda=1}^3 (a_v \cdot b_\lambda) a_\lambda^{-1} \right) + a_v \left(b_\mu - \sum_{\lambda=1}^3 (a_\mu \cdot b_\lambda) a_\lambda^{-1} \right) \\ = a_\mu b_v - a_v b_\mu + a_v b_\mu - a_\mu b_v = 0 \quad (\mu, v=1, 2, 3),$$

i. e. the necessary and sufficient conditions according to Pr 73, Pr 78 for the consistency of the system of vector equations (276) are satisfied. Then Pr 78 implies, that (276) has exactly one solution (140). This solution is (277) according to (254).

Pr 99. (131), (132), (158) imply: the system of vector equations (276) has exactly one solution (140), namely

$$(279) \quad r = \frac{1}{a_1 \times a_2 \cdot a_3} \sum_{v=1}^3 (a_{v+2} b_{v+1} - a_{v+1} b_{v+2}) a_v$$

with (147), (148).

Dm. Pr 98, Pr 61.

Pr 100. (131), (132), (167) imply: the system of vector equations (276) has exactly one solution (140), namely

$$(280) \quad r = \sum_{v=1}^3 a_v \times b_v.$$

Dm. Pr 21, Pr 41, Pr 98.

Pr 101. (131) (132), (166) imply: the system of vector equations (276) has exactly one solution (140), namely

$$(281) \quad r = \sum_{s=1}^3 (a_{s+2} b_{s+1} - a_{s+1} b_{s+2}) a_s.$$

Dm. Pr 99.

Pr 102. (131), (132), (189) imply: the system of vector equations (276) has exactly one solution (140), namely

$$(282) \quad r = \sum_{s=1}^3 (a_{s+1} b_{s+2} - a_{s+2} b_{s+1}) a_s.$$

Dm. Pr 99.

Pr 103. (131), (132), (140).

$$(283) \quad c_v \in V_F \quad (v=1, 2, 3)$$

imply: a necessary condition for the consistency of the system of vector equations

$$(284) \quad r \times a_v = b_v - c_v \quad (v=1, 2, 3)$$

is

$$(285) \quad a_\mu b_v + a_v b_\mu = a_\mu c_v + a_v c_\mu \quad (\mu, v=1, 2, 3).$$

Dm. Pr 73.

Pr 104. (131), (132), (158), (283), (285) imply: the system of vector equations (284) has exactly one solution (140), namely

$$(286) \quad r = \frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times b_v - \frac{1}{2} \sum_{v=1}^3 a_v^{-1} \times c_v.$$

Dm. Pr 78.

Pr 105. (131), (132), (140), (158), (283) imply: (285) is a sufficient condition for the consistency of the system of vector equations (284).

Dm. Pr 104.

Pr 106. (131), (132), (158), (283), (285) imply: the system of vector equations (284) has exactly one solution (140), namely (253) iff

$$(287) \quad a_\mu c_v = \frac{1}{2} (a_\mu b_v + a_v b_\mu) \quad (\mu, v=1, 2, 3).$$

Dm. Pr 104, Pr 62.

Pr 107. (131), (132), (158),

$$(288) \quad c_v = -\frac{1}{2} b_v + \frac{1}{2} \sum_{\lambda=1}^3 (a_\lambda b_\lambda) a_\lambda^{-1} \quad (v=1, 2, 3)$$

imply (285).

Dm. Pr 23 implies

$$(289) \quad a_\mu c_\nu + a_\nu c_\mu = \frac{1}{2} a_\mu \left(b_\nu + \sum_{\lambda=1}^3 (a_\lambda b_\nu) a_\lambda^{-1} \right)$$

$$+ \frac{1}{2} a_\nu \left(b_\mu + \sum_{\lambda=1}^3 (a_\lambda b_\mu) a_\mu^{-1} \right) = \frac{1}{2} (a_\mu b_\nu + a_\nu b_\mu)$$

$$+ \frac{1}{2} (a_\nu b_\mu + a_\mu b_\nu) = a_\mu b_\nu + a_\nu b_\mu \quad (\mu, \nu = 1, 2, 3).$$

Pr 108, (131), (132), (158), (283) imply: the system of vector equation (284) has exactly one solution (140), namely (253) iff (288).

Dm. Pr 105 — Pr 107, Pr 65, Pr 55.

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ВЕКТОРИ НА ГИБС В ТРИМЕРНО ВЕКТОРНО ПРОСТРАНСТВО НАД НАРЕДЕНО ПОЛЕ

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(РЕЗЮМЕ)

В някои предидущи работи на авторите [1] — [12] се подчертава важната роля, която векторите на Гибс играят в редица въпроси на линейната алгебра, линейната аналитична геометрия и аналитичната механика. Към тези въпроси спадат: изразите (47) и (55) за моментната ъгловая скорост $\dot{\omega}$ и моментното ъглово ускорение $\ddot{\epsilon}$ на твърдо тяло; изразът (88) за локалната скорост на точка на афинно тяло, където $\dot{\omega}$ е аналог на моментната ъгловая скорост в афинната кинематика; изразите (117), (118), определящи ротацията (114), която осъществява крайното преместване, определено от реперите (89), (90) с (91), (92); технически изящната възможност за аксиоматично обосноваване на кинематиката на афинните и твърдите тела в реалните и комплексните хилбертови пространства и т. н. Оказва се, че векторите на Гибс работят в значително по-общи ситуации, отколкото в реалното тримерно векторно пространство, например в тримерно векторно пространство над произволно наредено поле. В настоящата работа са дадени основните положения на алгебрата им в такова пространство.