A CLASS OF COMPACT ABELIAN GROUPS

Dikran Dikranjan, Ivan Prodanov

In this paper we consider compact Abelian groups G such that each non-zero closed subgroup of G contains non-zero periodic elements of G. It turns out that they possess many properties of the usual standard tori. We call them exotic tori.

Section 1 is purely algebraical, and contains the main properties of the character groups of the exotic tori. They are called strongly non-divisible groups and may be characterised in various ways. Some of them are exposed in Theorem 1.4. The structure of those groups is studied in Propositions 1.5, 1.6 and in Theorem 1.9.

Section 2 is devoted to the exotic tori. Some of their properties are obtained by a translation of the corresponding ones in Section 1, using the Pontrjagin's duality. Characteristic conditions are given in Theorems 2.4* and 2.4.

The exotic tori are closely related to minimal and totally minimal group topologies. A topological group G is called *minimal* (and the topology of G is said to be *minimal*) if G is Hausdorff, and every group topology on G which is coarser than the topology of G is non-Hausdorff. A topological group G is called *totally minimal*, if each quotient group G/H with respect to a closed normal subgroup H is minimal. The first examples of non-compact minimal groups may be found in [3] and [6]. Examples of non-compact totally minimal groups are given in [2].

In Section 3 we apply exotic tori to study minimal and totally minimal topologies on periodic groups. Theorem 3.3 for example states that a periodic divisible group G admits a minimal precompact group topology if and only if $G=(\mathbf{Q}/\mathbf{Z})^n$ $(n=1, 2, \cdots)$. The precompact minimal group topologies on those groups are totally minimal and are induced by the embeddings into the connected exotic tori.

In general, the notations follow those of [5]. If G is an Abelian group, by $T(G)$ we denote the periodic subgroup of G, by $S(G)$ — the socle of G, i. e. the subgroup of G generated by the elements of G with prime period, and by $r(G)$ — the rank of G.

1. STRONGLY NON-DIVISIBLE ABELIAN GROUPS

An Abelian group X is called *strongly non-divisible*, if for each proper subgroup Y of X the quotient group X/Y is non-divisible. Obviously that condition means that there exists a natural n with $nX+Y\neq X$.

Clearly every strongly non-divisible group is reduced, but the converse is not true. For example, if p is a prime the group

$$\mathbf{Q}_p=\left\{\frac{m}{p^n}:\ m\in\mathbf{Z},\ n\in\mathbf{N}\right\}$$

is reduced, while the quotient group $\mathbf{Q}_p/\mathbf{Z}=\mathbf{Z}\,(p^\infty)$ is divisible. The group $\bigoplus_{k=1}^{\infty}\mathbf{Z}/p^k\,\mathbf{Z}$ is periodic, reduced, and not strongly non-divisible.

The following proposition demonstrates an elementary property of the strongly non-divisible groups. It is proved in more general form in [1]. For completeness we give a proof.

1.1. Proposition. If the sequence

(1) $$0\to X'\to X\to X''\to 0$$

of Abelian groups and homomorphisms is exact, the group X is strongly non-divisible if and only if X' and X'' are strongly non-divisible.

Proof. The necessity is evident. Let X' and X'' be strongly non-divisible. Assume there exists an epimorphism $\varphi: X\to D$, where D is non-zero divisible group. If $X'\not\subset\ker\varphi$, we consider the restriction $\varphi: X'\to D$. φ is not an epimorphism, since X' is strongly non-divisible. Hence the divisible group $D/\varphi(X')$ is non-zero. Taking the composition

$$X\xrightarrow{\varphi}D\to D/\varphi(X')$$

instead of φ, we may assume $X'\subset\ker\varphi$. The exactness of (1) gives an epimorphism ψ: $X''\to D$, which is a contradiction, since X'' is strongly non-divisible. Therefore X is strongly non-divisible. Q.E.D

1.2. Corollary. Let $X_1, X_2, \cdots, X_n$ be Abelian groups. Then the group $X_1\oplus X_2\oplus\cdots\oplus X_n$ is strongly non-divisible if and only if each X_k is strongly non-divisible ($k=1, 2, \cdots, n$).

1.3. Corollary. Every finitely generated Abelian group is strongly non-divisible.

It suffices to show that the group $\mathbf{Z}$ is strongly non-divisible, which is obvious.

The following theorem contains a few equivalent forms of the definition of a strongly non-divisible group.

1.4. Theorem. Let X be an Abelian group. Then the following five conditions are equivalent:

i) X is strongly non-divisible;

ii) there does not exist an epimorphism of the form $\varphi: X\to\mathbf{Z}\,(p^\infty)$ $(p\in\mathbf{P})$;

iii) the rank of X is finite and for every free subgroup A of X with $r(A)=r(X)$ holds

(2) $$X/A = \bigoplus_{p \in P} T_p,$$

where for each $p \in P$ there exists a natural k with

(3) $$p^k T_p = 0;$$

iv) the rank of X is finite and there exists a free subgroup A of X with $r(A)=r(X)$ such that (2) and (3) are fulfilled;

v) each proper subgroup of X is contained in a maximal proper subgroup.

Proof. i) $\Rightarrow$ ii). Trivial.

ii) $\Rightarrow$ iii). Let us assume that the rank of X is infinite. If $\{X_n\}_{n=1}^{\infty}$ is an infinite independent system in X, we obtain an epimorphism $\varphi: H \to \mathbf{Z}(p^\infty)$ setting $\varphi(X_n) = \frac{1}{p^n} + \mathbf{Z}$ $(n=1, 2, \dots)$, where H is the subgroup generated by $\{x_n\}_{n=1}^{\infty}$. Since $\mathbf{Z}(p^\infty)$ is divisible, we may extend φ to X, which contradicts ii). Therefore the rank of X is finite.

Let A be a free subgroup of X with $r(A)=r(X)$. Then X/A is periodic, hence there exists a representation (2), where T_p is a p-group $(p \in \mathbf{P})$. We have only to prove that for each $p \in \mathbf{P}$ there exists $k \in \mathbf{N}$ with (3). For this purpose it is sufficient to know that the group T_p^* of the characters of T_p is periodic. In fact, the compactness of T_p^* implies the existence of a common period of the elements of T_p^*. Then the elements of T_p have the same common period, since the characters separate the points of T_p. Thus we obtain (3).

Let us assume that the group T_p^* is not periodic. Then there exists a non-periodic character $\chi: T_p \to \mathbf{R}/\mathbf{Z}$. Obviously $\chi(T_p) \subset \mathbf{Z}(p^\infty)$, and the non-periodicity of χ implies $\chi(T_p) = \mathbf{Z}(p^\infty)$. We extend the epimorphism $\chi: T_p \to \mathbf{Z}(p^\infty)$ to X/A. Then the composition

$$X \to X/A \xrightarrow{\chi} \mathbf{Z}(p^\infty)$$

is an epimorphism which contradicts ii).

iii) $\Rightarrow$ iv). Obvious.

iv) $\Rightarrow$ v). Let Y be a proper subgroup of X. If $Y+A=X$, then

$$X/Y = (Y+A)/Y \cong A/(A \cap Y)$$

is a finitely generated group. Hence there exists a maximal proper subgroup H of X/Y. The inverse image of H is a maximal proper subgroup of X containing Y. Consider now the case $Y+A \neq X$, and let

$$\sigma: X \to X/A = \bigoplus_{p \in \mathbf{P}} T_p$$

be the canonical epimorphism. Then $\sigma(Y) \neq \bigoplus_{p \in \mathbf{P}} T_p$, hence it suffices to show that the group

$$\Big(\bigoplus_{p \in \mathbf{P}} T_p\Big)/\sigma(Y) = \bigoplus_{p \in \mathbf{P}_k} T'_p$$

possesses maximal subgroups. By (3), $p^k T'_p=0$ holds. We prove that every T'_p possesses maximal subgroups. In fact, for each character $\chi: T'_p \to \mathbf{R}/\mathbf{Z}$ we have $\chi(T'_p)\subset \mathbf{Z}/p^k\mathbf{Z}$. Since the latter group is finite, it has maximal subgroups.

v) $\Rightarrow$ *i*). Let Y be a proper subgroup of X. We have to verify that the group X/Y is not divisible. Take a maximal subgroup M of X which contains Y. Then X/M is cyclic, hence finitely generated and non-divisible according to Corollary 2. Therefore X/Y is also non-divisible. **Q.E.D.**

The following proposition shows that the strongly non-divisible groups can be approximated by direct sums of free groups and periodic groups. We shall denote from now on by $p_1, p_2, \dots, p_k, \dots$ the sequence of the primes (in an arbitrary order).

1.5. Proposition. An Abelian group X of rank n is strongly non-divisible if and only if there is a representation

$$X=\bigcup_{m=0}^{\infty} X_m, \tag{4}$$

where the group X_0 is free and $r(X_0)=n$,

$$X_0\subset X_1\subset X_2\subset \cdots \subset X_m\subset \cdots \tag{5}$$

is an increasing sequence of subgroups and for each natural m there exists a non-negative integer k_m, such that

$$p_m^{k_m} X_m\subset X_{m-1} (m=1, 2, \dots). \tag{6}$$

Proof. Let X be a strongly non-divisible group. By iv) in Theorem 1.4 there exists a free subgroup A of X with $r(A)=n$ for which (2) and (3) hold. For an arbitrary non-negative integer m we denote by X_m the inverse image of $\bigoplus_{\mu=1}^{m} T_{p_\mu}$ under the canonical homomorphism $X\to\bigoplus_{p\in P} T_p$. It is easy to verify that (4), (5) and (6) hold.

Conversely, if the group X has a representation satisfying (4), (5) and (6), the group X_0 is free with $r(X_0)=n$ and

$$X/X_0=\bigoplus_{m=1}^{\infty} T_{p_m},$$

where $p_m^{k_m} T_{p_m}=0$ $(m=1, 2, \dots)$, since $p_1^{k_1} p_2^{k_2}\cdots p_m^{k_m} X_m\subset X_{m-1}$ by (6). Now we apply Theorem 1.4, iv). **Q.E.D.**

Let X be a strongly non-divisible group. Each representation (4) with (5) and (6) is called *canonical*.

The following two propositions describe two properties of canonical representations.

1.6. Proposition. Let X be a strongly non-divisible group of rank n and (4) be a canonical representation of X. Then each of the groups X_m $(m=1, 2, \dots)$ is a direct sum of a free subgroup of rank n and a periodic subgroup of X with period $p_1^{k_1}\cdots p_m^{k_m}$, where $k_1, k_2, \dots$ are determined in (6).

Proof. We have $r(X_0)=r(X)=n$. Now (5) implies $r(X_m)=n$ $(m=1, 2, \dots)$. By (6), $p_1^{k_1}\cdots p_m^{k_m} X_m \subset X_0$. Since X_0 is free, the periodic elements of X_m have a common period $\tau=p_1^{k_1}\cdots p_m^{k_m}$. Thus we obtain the exact sequence

$$0 \to T(X_m) \to X_m \xrightarrow{\tau} \tau X_m \to 0$$

which splits, since the group τX_m is free. **Q.E.D.**

1.7. Proposition. Let X be a strongly non-divisible torsion free group, and (4) be a canonical representation of X. Then the groups X_m $(m=1, 2, \dots)$ are free and each of the quotient groups X_m/X_{m-1} $(m=1, 2, \dots)$ is a finite p_m-group.

Proof. Obvious.

The following lemma is a step to the main property of the canonical representations.

1.8. Lemma. Let X be a strongly non-divisible group and (4) be a canonical representation of X. If for $x \in X$ there exist natural numbers N and m such that

$$(p_1 p_2 \cdots p_m)^N x \in X_m, \tag{7}$$

then $x \in X_m$.

Proof. By (4), there exists t with $x \in X_t$. If $t \leq m$, the statement is proved. Assume $t>m$. Then there exist integers u and v such that

$$1=u(p_1 p_2 \cdots p_m)^N + v\, p_{m+1}^{s_{m+1}} p_{m+2}^{s_{m+2}} \cdots p_t^{s_t}.$$

Then

$$x=u(p_1 p_2 \cdots p_m)^N x + v(p_{m+1}^{s_{m+1}} p_{m+2}^{s_{m+2}} \cdots p_t^{s_t})x. \tag{8}$$

From (6) and $x \in X_t$ it follows

$$v p_{m+1}^{s_{m+1}} p_{m+2}^{s_{m+2}} \cdots p_t^{s_t} x \in X_m.$$

Hence $x \in X_m$ in account of (7) and (8). **Q.E.D.**

1.9. Theorem. Let X be a strongly non-divisible group, and (4) and

$$X=\bigcup_{m=0}^{\infty} X'_m \tag{9}$$

be canonical representations of X. Then for each sufficiently large m we have

$$X_m = X'_m. \tag{10}$$

Proof. It is enough to see that for each sufficiently large m we have $X_m \subset X'_m$. Since the group X'_0 is of maximal rank, and X_0 is finitely generated, there are natural l and s with

$$(p_1 p_2 \cdots p_l)^s X_0 \subset X'_0.$$

Now (5) implies

$$p_1^{k_1+s}\, p_l^{k_l+s}\, p_{l+1}^{k_l+1} \cdots p_m^{k_m}\, X_m \subset X_0' \subset X_m' \, (m>l).$$

Hence by Lemma 1.8 $X_m \subset X_m'$ $(m>l)$. **Q.E.D.**

1.10. Corollary. If X and Y are isomorphic strongly non-divisible groups, and (4) and

$$Y=\bigcup_{m=0}^{\infty} Y_m \tag{11}$$

are canonical representations of X and Y respectively, the groups X_m/X_{m-1} and Y_m/Y_{m-1} are isomorphic for each sufficiently large m.

Let X be a strongly non-divisible torsion free group, and (4) be a canonical representation of X. By Proposition 1.7 for each natural m there is a non-negative s_m such that $|X_m/X_{m-1}|=p_m^{s_m}$. Thus we find an infinite sequence

$$s_1, s_2, \dots, s_m, \dots \tag{12}$$

We shall call (12) a *determinant of the canonical representation* (4). By Corollary 1.10, the determinants of every two canonical representations coincide for sufficiently large m. Thus to every strongly non-divisible torsion free group X corresponds an equivalence class of sequences (12) which coincide for sufficiently large m. We shall call this class (and also each sequence (12)) a *determinant of X*. Obviously the determinants of isomorphic strongly non-divisible torsion free groups coincide.

The following two propositions contain properties of the determinant of a subgroup and a quotient group respectively.

1.11. Proposition. Let X be a strongly non-divisible torsion free group and X' be a subgroup of X. If (12) is a determinant of X, and $s_1', s_2', \dots, s_m' \dots$ — a determinant of X', we have $s_m' \leq s_m$ for each sufficiently large m.

Proof. If (4) is a canonical representation of X, the sequence

$$X_m' = X_m \cap X' \quad (m=0, 1, 2, \dots)$$

is a canonical repesentation of X'. Now the statement follows from the obvious inclusion

$$X_m'/X_{m-1}' \subset X_m/X_{m-1} \quad (m=1, 2, \dots).$$

Q.E.D.

1.12. Proposition. Let X be a strongly non-divisible torsion free group, and X' be a subgroup of X such that X/X' is torsion free. If (12) is a determinant of X, and $s_1'', s_2'', \dots, s_m'', \dots$ — a determinant of X/X', we have $s_m'' \leq s_m$ for each sufficiently large m.

Proof. If (4) is a canonical representation of X, the sequence

$$X_m'' = X_m/(X_m \cap X') \; (m=0, 1, 2, \dots)$$

is a canonical representation of X/X'. Now the statement follows from the isomorphism

(13) $$X''_m/X''_{m-1} \cong X_m/(X_{m-1}+X_m \cap X') \quad (m=1, 2, \ldots)$$

and from the fact that the right side of (13) is a quotient group of X_m/X_{m-1}. Q.E.D.

Propositions 5 and 6 show that strongly non-divisible groups are closely related to the free groups of finite rank. The following examples point out some differences.

Example 1. For each natural n there exists a continuum of non-isomorphic strongly non-divisible torsion free groups of rank n. Obviously, it is enough to show that every sequence (12) of non-negative integers is a determinant of a strongly non-divisible torsion free group X of rank n, and that is straightforward.

Example 2. The periodic subgroup of a strongly non-divisible group is not always a direct summand. Let X' be the subgroup of $\mathbf{Q}$, generated by the numbers $x_k=\frac{1}{p_1^2 p_1^2 \cdots p_k^2}$ ($k=1, 2, \ldots$) and $T=\bigoplus_{p \in P}(\mathbf{Z}/p\mathbf{Z})$. We show first that

(14) $$\mathrm{Ext}^1(X', T) \neq 0.$$

rom the exact sequence

$$0 \to T \xrightarrow{i} \mathbf{Q}/\mathbf{Z} \xrightarrow{u} \mathbf{Q}/\mathbf{Z} \to 0,$$

where i is the canonical embedding, and u is the mutiplication of the p-th component by p for each $p \in \mathbf{P}$, we obtain the exact sequence

$$0 \to \mathrm{Hom}(X', T) \xrightarrow{i_*} \mathrm{Hom}(X', \mathbf{Q}/\mathbf{Z}) \xrightarrow{u_*} \mathrm{Hom}(X', \mathbf{Q}/\mathbf{Z}) \to \mathit{Ext}'(X'\, T).$$

To prove (14) it is enough to show that u_* is not an epimorphism. Consider the homomorphism

$$\varphi : X' \to X'/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$$

and assume that there exists a homomorphism $f: X' \to \mathbf{Q}/\mathbf{Z}$ with $uf=\varphi$. Then $uf(1)=\varphi(1)=\mathbf{Z}$. Hence there are $s \in \mathbf{Z}$ and $m \in \mathbf{N}$ with

(15) $$f(1)=\frac{s}{p_1 p_2 \cdots p_m}+\mathbf{Z}.$$

On the other hand, the equalities $uf(x_{m+1})=\varphi(x_{m+1})=x_{m+1}+\mathbf{Z}$ imply

(16) $$f(x_{m+1})=\frac{v}{p_1^3 p_2^3 \cdots p_{m+1}^3}+\frac{t}{p_1 p_2 \cdots p_l}+\mathbf{Z} \quad (l \in \mathbf{N}),$$

where $v, t \in \mathbf{Z}$, and $(v, p_1 p_2 \cdots p_{m+1})=1$. From (15), (16) and $(p_1 p_2 \cdots p_{m+1})^3 \times x_{m+1}=1$ it follows

$$\frac{s}{p_1 p_2 \cdots p_m}+\mathbf{Z}=\frac{v}{p_1 p_2 \cdots p_{m+1}}+\frac{t'}{p_{m+2} \cdots p_l}+\mathbf{Z} \quad (t' \in \mathbf{Z})$$

which is impossible, since p_{m+1} does not divide v. Therefore a homomorphism f with $uf=\varphi$ does not exist. This proves (14). From (14) it follows that there is an Abelian group X such that the sequence

$$0 \to T \to X \to X' \to 0$$

is exact and does not split. Since the groups T and X' have the property iv) of Theorem 1.4, they are strongly non-divisible. Proposition 1.1 now implies that the group X is also strongly non-divisible.

Example 3. If X is a strongly non-divisible group such that in the representation $T(X)=\bigoplus_{p\in P} T_p$ only finitely many of the summands T_p are different from 0, $T(X)$ is a direct summand of X. Indeed, we have the exact sequence

$$0\to T(X)\to X\xrightarrow{\lambda} X'\to 0, \tag{17}$$

where the group $X'=X/T(X)$ is strongly non-divisible and torsion free. Let $X'=\bigcup_{m=0}^{\infty} X'_m$ be a canonical representation of X', $t\neq 0$ be a period of $T(X)$, and for $m\geqq l$ the prime p_m does not divide t. Choose a homomorphism $\mu_l : X_l'\to X$ such that $\lambda\mu_l$ is the identity of X'_l. We show that μ_l may be extended to a homomorphism $\mu_{l+1} : X'_{l+1}\to X$ such that $\lambda\mu_{l+1}$ is the identity of X'_{l+1}. Let $x'_1, x'_2, \ldots, x'_n$ be a base of X'_{l+1} such that $p_{l+1}^{s_1} x'_1$, $p_{l+1}^{s_2} x'_2, \ldots, p_{l+1}^{s_n} x'_n$ be a base of X'_l. Since λ is an epimorphism, there are elements $x_1, x_2, \ldots, x_n$ of X with $\lambda(x_\lambda)=x'_\nu$ $(\nu=1, 2, \ldots, n)$. At the same time

$$\lambda(p_{l+1}^{s_\nu} x_\nu-\mu_l(p_{l+1}^{s_\nu} x'_\nu))=p_{l+1}^{s_\nu} x'_\nu-p_{l+1}^{s_\nu} x'_\nu=0.$$

Hence

$$p_{l+1}^{s_\nu} x_\nu-\mu_l(p_{l+1}^{s_\nu} x'_\nu)=y_\nu\in T(X).$$

Since p_{l+1} does not divide t, there is a $z_\nu\in T(X)$ with

$$p_{l+1}^{s_\nu}(x_\nu-z_\nu) = \mu_l(p_{l+1}^{s_\nu} x'_\nu).$$

Now it is clear that the equalities

$$\mu_{l+1}(x'_\nu)=x_\nu-z_\nu \quad (\nu=1, 2, \ldots, n)$$

define a desirable extention of μ_l. In the same way we extend μ_{l+1} to a homomorphism μ_{l+2} $X'_{l+2}\to X$, and so on. Thus by induction we receive a homomorphism $\mu : X'\to X$ such that λ_μ is the identity of X'. Hence (17) splits, and so $T(X)$ is a direct summand.

Example 4. There exists a strongly non-divisible torsion free group X and a subgroup Y of X such that X/Y is torsion free and Y is not a direct summand in X. Indeed, let X' be as in Example 2, and Y be the subgroup of $\mathbf{Q}$, generated by the numbers

$$y_k=\frac{1}{p_1 p_2 \cdots p_k} \quad (k=1, 2, \ldots).$$

It follows from Proposition 1.5 that the groups X' and Y are strongly non-divisible. Using the exact sequence

$$0\to Y\to \mathbf{Q}\to \mathbf{Q}/Y\to 0,$$

as in Example 2 we show that $\operatorname{Ext}^1(X', Y)\neq 0$. Now the argument finishes as in Example 2.

2. EXOTIC TORI

A compact Abelian graup G is called *exotic torus* if each non-zero closed subgroup of G contains non-zero periodic elements of G.

Clearly, each n-dimensional standard torus is an exotic torus.

2.1. Proposition. A compact Abelian group G is an exotic torus if and only if the group G^* of the continuous characters of G is strongly non-divisible.

Proof. It follows immediately from the Pontrjagin's duality that an Abelian group X is non-divisible if and only if the group X^* contains non-zero periodic elements. Therefore X is strongly non-divisible if and only if each non-zero closed subgroup of X^* contains non-zero periodic elements. **Q.E.D.**

This proposition permits us to transfer mutatis mutandis the content of Section 1 to exotic tori. For the sake of completeness we lay down a few of the statements found in this way.

2.1*. Proposition. If the sequence

$$0 \to G'' \to G \to G' \to 0$$

of compact Abelian groups and continuous homomorphisms is exact, then G is an exotic torus if and only if G' and G'' are.

2.4*. Theorem. For an arbitrary compact Abelian group G the following five conditions are equivalent:

i)* G is an exotic torus;

ii)* for each prime p the group G contains no copy of $\mathbf{Z}_p$;

iii)* the space G is finite dimensional, and if $n = \dim G$ for each continuous epimorphism

$$\lambda : G \to \mathbf{T}^n \tag{1}$$

we have

$$\ker \lambda = \prod_{p \in \mathbf{P}} G_p, \tag{2}$$

where G_p is a compact p-group $(p \in \mathbf{P})$;

iv)* the space G is finite dimensional, and if $n = \dim G$, there is a continuous epimorphism (1) with (2);

v)* each non-zero closed subgroup of G contains a minimal closed non-zero subgroup of G.

2.5*. Proposition. A compact Abelian group G of dimension n is an exotic torus if and only if there is a representation of G as a projective imit

$$G = \varprojlim_{m} G_m \tag{3}$$

lsuch that $G_0 = \mathbf{T}^n$, the continuous homomorphisms σ_m in the projective system

$$G_0 \xleftarrow{\sigma_1} G_1 \xleftarrow{\sigma_2} G_2 \leftarrow \cdots \xleftarrow{\sigma_m} G_m \leftarrow \cdots \tag{4}$$

are epimorphisms, and $\ker \sigma_m$ is a compact p_m-group for each natural m.

We call a representation (3) with the properties described in Proposition 2.5* a *canonical representation of the exotic torus G*.

2.6*. Proposition. If G is an exotic torus with $\dim G = n$ and (3) is a canonical representation of G, then each of the groups G_m is a product of $\mathbf{T}^n$ and a periodic compact group with a period

$$p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} \quad (k_\mu \in \mathbf{N},\ \mu=1, 2, \ldots, m).$$

2.7*. Proposition. If G is a connected exotic torus with $n=\dim G$, and (3) is a canonical representation of G, then each of the groups G_m coincides with $\mathbf{T}^n$, and $\ker \sigma_m$ is a finite p_m-group.

Example 1*. For each natural n there is a continuum of non-isomorphic connected n-dimensional exotic tori.

Example 2*. The connected component of 0 in an exotic torus G is not always a topological direct multiplicand of G.

2.2. Proposition. Let G be an exotic torus and G_1 be the connected component of 0. Then

$$G/G_1 = \prod_{p\in \mathbf{P}} T_p,$$

where T_p is a compact p-group ($p \in \mathbf{P}$). Algebraically G_1 is a direct summand in G.

Proof. It follows from Proposition 2.1 that the group $X=G^*$ is strongly non-divisible. Let $T(X)=\bigoplus_{p\in\mathbf{P}} T_p$. By property iii) in Theorem 1.4, for each $p \in \mathbf{P}$ there is a natural k with $p^k T_p=0$. From the exact sequence

$$0 \to \bigoplus_{p\in\mathbf{P}} T_p \to X \to X' \to 0,$$

where X' is a strongly non-divisible torsion free group we obtain the exact sequence

$$0 \leftarrow \prod_{p\in\mathbf{P}} T_p^* \leftarrow G \leftarrow X'^* \leftarrow 0,$$

where the group X'^* is connected, since X' is torsion free. It is not difficult to see that X'^* is the connected component of 0 in G. On the other hand, the group $G_1=X'^*$ is divisible, and hence G_1 is a direct summand in G. **Q.E.D.**

2.3. Proposition. For every exotic torus G the periodic subgroup $T(G)$ is dense in G.

Proof. By Proposition 2.1, the group $X=G^*$ is strongly non-divisible. It follows from the Pontrjagin's duality that the statement will be proved, if we show that the periodic characters of X separate the points of X.

For each $\xi \in X$ with $\xi \neq 0$ we have to verify that there is a periodic character $\chi: X \to \mathbf{T}^1$ with $\chi(\xi) \neq 0$. Let $n=r(X)$, and A be a free subgroup of X with $r(A)=n$ and $\xi \overline{\in} B$. By the Property iii) in Theorem 1.4, we have an exact sequence

$$0 \to A \to X \xrightarrow{\sigma} \bigoplus_{p\in\mathbf{P}} T_p \to 0,$$

where for each $p \in \mathbf{P}$ there is a natural k with

$$p^k T_p = 0. \tag{5}$$

From $\xi \bar{\in} A$ it follows $\sigma(\xi) \neq 0$. Therefore, we only have to show that the periodic characters separate the points of T_p. But it is well-known that the characters separate the points of every discrete Abelian group, and on the other hand it follows from (5) that the characters of T_p are periodic. **Q.E.D.**

The following theorem contains some other characteristic properties of the exotic tori.

2.4. Theorem. For every compact Abelian group G the following three conditions are equivalent:

i) G is an exotic torus;

ii) the group $T(G)$ is dense and totally minimal;

iii) the group $T(G)$ is dense and minimal.

Proof. i) $\rightarrow$ ii). By Proposition 2.3, the group $T(G)$ is dense. Let H be a non-zero closed subgroup of G. Then H is an exotic torus. Again by Proposition 2.3, the group $T(G) \cap H = T(H)$ is dense in H. Now Theorem 1 in [2] shows that $T(G)$ is totally minimal.

ii) $\Rightarrow$ iii). Obvious.

iii) $\Rightarrow$ i). By Theorem 2 in [6], each non-zero closed subgroup of G intersects $T(G)$ in a non-trivial way. Hence G is an exotic torus. **Q.E.D.**

Let G be a connected exotic torus. Then the group $X = G^*$ is strongly non-divisible and torsion free. Therefore we may form the determinant

$$s_1, s_2, \dots, s_m, \dots \tag{6}$$

of X. We shall call (6) (and the equivalence class of all the sequences coinciding with (6) for sufficiently large m) a *determinant* of G. It is easy to see that if (6) is a determinant of G and (3) is a canonical repesentation of G, then for each sufficiently large m the number of the elements of ker σ_m coincides with $p_m^{s_m}$. If two connected exotic tori are algebraically and topologically isomorphic, then their determinants (as equivalence classes) coincide. If dim $G=1$ the opposite is also true, but if dim $G>1$ the situation is more complicated. A connected exotic torus G with dim $G=n$ is isomorphic with T^n if and only if $s_m=0$ for all but a finite number of m's.

2.5. Proposition. If G is a connected exotic torus, and in (6) infinitely many s_m are zero, then for each connected closed subgroup H of G the socle $S(H)$ is dense in H.

Proof. First we consider the case $H=G$. Using the Pontrjagin's duality, we see that the statement is equivalent to the following one.

If X is a strongly non-divisible torsion free group, and in the determinant (6) of X infinitely many s_m are zero, then $S(X^*)$ separates the points of X.

To show this, choose an arbitrary $\xi \in X$ with $\xi \neq 0$. Let $n = r(X)$, and A be a free subgroup of X with $r(A) = n$, and $\xi \bar{\in} A$. Consider the quotient epimorphism

$$\lambda : X \to X/A = \bigoplus_{p \in P} T_p. \tag{7}$$

Obviously $\lambda(\xi) \neq 0$. Let the period of $\lambda(\xi)$ be m. Then $m\xi \in A$. Therefore there exist a base $x_1, x_2, \dots, x_n$ of A and $t \in \mathbf{Z}$, $t \neq 0$ such that $m\xi = t x_1$. Now choose a prime q in such a way that $(q, m) = 1$, $(q, t) = 1$, $T_q = 0$, and denote by A' the group generated by $qx_1, x_2, \cdots, x_n$. Since $A/A' = \mathbf{Z}/q\mathbf{Z}$, it follows by (7)

$$X/A' = (\bigoplus_{p\in P} T_p) \oplus (\mathbf{Z}/q\mathbf{Z}).$$

Let

$$\lambda' : X \to X/A' = (\bigoplus_{p\in P} T_p) \oplus (\mathbf{Z}/q\mathbf{Z})$$

be the canonical quotient epimorphism, and

$$\mu : (\bigoplus_{p\in P} T_p) \oplus (\mathbf{Z}/q\mathbf{Z}) \to \mathbf{Z}/q\mathbf{Z}$$

be the corresponding projection. Then $\mu\lambda' : X \to \mathbf{Z}/q\mathbf{Z}$ is a character from $S(X^*)$. Therefore the statement will be proved, if we show that

$$\mu\lambda'(\xi) \neq 0. \tag{8}$$

From $(t, q) = 1$ it follows $tx_1 \overline{\in} A'$. Hence

$$\lambda'(m\xi) = \lambda'(tx_1) \neq 0. \tag{9}$$

On the other hand

$$q\lambda'(m\xi) = \lambda'(tqx_1) = 0, \tag{10}$$

ince $qx_1 \in A'$. As for each $p \neq q$ we have $qT_p = T_p$, (9) and (10) imply $\mu\lambda'(m\xi) \neq 0$. Now (8) is obvious and the statement is proved for $H = G$.

Let now H be a connected closed subgroup of G. Then H is a connected exotic torus. By Proposition 1.12, infinitely many members of the determinant of H are zero. Now the above proof is aplicable. **Q.E.D.**

3. MINIMAL PRECOMPACT TOPOLOGIES ON PERIODIC ABELIAN GROUPS

Recall that a Hausdorff topological group G is called *precompact*, if the completion $\hat{G}$ of G is compact.

3.1. Proposition. Let G be a periodic minimal precompact Abelian group. Then the completion $\hat{G}$ is an exotic torus.

Proof. Since $G \subset T(\hat{G})$ and G is minimal, $T(\hat{G})$ is minimal and dense in $\hat{G}$. By Theorem 2.4, $\hat{G}$ is an exotic torus. **Q E.D.**

3.2. Corollary. The group G^* of the continuous characters of a periodic minimal precompact Abelian group G is strongly non-divisible.

Indeed, $G^* = (\hat{G})^*$, and we apply Proposition 2.1.

3.3. Theorem. Let G be a periodic divisible Abelian group endowed with a minimal topology. Then $G = (\mathbf{Q}/\mathbf{Z})^n$ for a non-negative integer n, and G is totally minimal.

Proof. By Proposition 3.1, the completion $\hat{G}$ is an exotic torus. According to Property iv)* from Theorem 2.4* there exist a non-negative integer n and an exact sequence

(1) $$0 \to \prod_{p \in \mathbf{P}} G_p \to \hat{G} \xrightarrow{\lambda} \mathbf{T}^n \to 0,$$

where the groups G_p are compact, and for each $p \in \mathbf{P}$ there exists a non-negative integer k_p with $p^{k_p} G_p = 0$.

First we prove that

(2) $$\lambda(T(\hat{G})) = T(\mathbf{T}^n) = (\mathbf{Q}/\mathbf{Z})^n.$$

Suppose $\eta \in T^n$ and $k\eta = 0$. Then $\eta = \lambda(x)$ for a $x \in \hat{G}$ with $kx \in \prod_{p \in \mathbf{P}} G_p$, since the sequence (1) is exact. Let $k = p_1^{\alpha_1} \dots p_s^{\alpha_s}$. Choose v and w in $\prod_{p \in \mathbf{P}} G_p$ such that for their coordinates hold

$$v_p = \begin{cases} (kx)_p & \text{for } p = p_1, \dots, p_s, \\ 0 & \text{for } p \in \mathbf{P} \setminus \{p_1, \dots, p_s\}, \end{cases}$$

and

$$w_p = \begin{cases} 0 & \text{for } p = p_1, \dots, p_s, \\ (kx)_p & \text{for } p \in \mathbf{P} \setminus \{p_1, \dots, p_s\}. \end{cases}$$

Clearly, there exists $w_1 \in \prod_{p \in \mathbf{P}} G_p$ with $kw_1 = w$. Since $kx = v + w$, we have $k(x - w_1) = v$, and hence $x - w_1$ is periodic. This proves (2), because $\eta = \lambda(x) = \lambda(x - w_1)$.

On the other hand, $T(\prod_{p \in \mathbf{P}} G_p) = \bigoplus_{p \in \mathbf{P}} G_p$, and from (1) and (2) we obtain the exact sequence

(3) $$0 \to \bigoplus_{p \in \mathbf{P}} G_p \to T(\hat{G}) \to (\mathbf{Q}/\mathbf{Z})^n \to 0.$$

We show now that

(4) $$T(\hat{G}) = G.$$

Obviously $G \subset T(\hat{G})$. Suppose $x \in T(\hat{G})$, then

$$q_1 \cdots q_m x = 0,$$

where $q_1, q_2, \dots, q_m$ are primes, not necessarily different. We prove $x \in G$ by induction. Let $m = 1$. Then the cyclic group generated by x is simple, and must intersect G in a non-trivial way, by Theorem 2 in [6]. Hence $x \in G$. Let now the statement be true for $m-1$. Then $q_1 \cdots q_{m-1}(q_m x) = 0$, and so $q_m x \in G$ by the inductive hypothesis. Since the group G is divisible, there exists $y \in G$ such that $q_m x = q_m y$. Hence $q_m(x-y)$, and so $x - y \in G$. Therefore $x \in G$, since $y \in G$. This prove (4).

From (3) and (4) we obtain the exact sequence

(5) $$0 \to \bigoplus_{p \in \mathbf{P}} G_p \to G \xrightarrow{\lambda} (\mathbf{Q}/\mathbf{Z})^n \to 0.$$

Since G is periodic and divisible, we have

(6) $$G=\bigoplus_{p\in\mathbf{P}}\mathbf{Z}(p^\infty)^{(\alpha_p)}.$$

On the other hand

(7) $$(\mathbf{Q}/\mathbf{Z})^n=\bigoplus_{p\in\mathbf{P}}(\mathbf{Z}(p^\infty))^n.$$

Consider the p-subgroups in (5). From (6) and (7) we obtain the exact sequence

(8) $$0\to G_p\to(\mathbf{Z}(p^\infty))^{(\alpha_p)}\xrightarrow{\lambda_p}(\mathbf{Z}(p^\infty))^n\to 0.$$

The exactness of (8) and $p^{k_p}G_p=0$ imply $\alpha_p=n$. Therefore $G\cong(\mathbf{Q}/\mathbf{Z})^n$.

On the other hand, by Theorem 2.4 and (4), G is totally minimal. **Q.E.D.**

3.4. Corollary. Let G be a connected exotic torus, and $n=\dim G$. Then $T(G)=(\mathbf{Q}/\mathbf{Z})^n$.

Indeed, by Theorem 2.4, the group $T(G)$ is minimal. On the other hand G is divisible, since G^* is torsion free. Hence $T(G)$ is also divisible.

3.5. Corollary. Let G be an exotic torus, and $n=\dim G$. Then $T(G)=(\mathbf{Q}/\mathbf{Z})^n\oplus(\bigoplus_{p\in\mathbf{P}}T_p)$, where T_p is a compact p-group $(p\in\mathbf{P})$.

Indeed, if G_1 is the component of 0 in G, then by Prosition 2.2 $G=G_1\oplus\prod_{p\in\mathbf{P}}T_p$, where T_p is a compact p-group. Clearly, for each $p\in\mathbf{P}$ there exists k_p with $p^{k_p}T_p=0$. Then

$$T(G)=T(G_1)\oplus T\Big(\prod_{p\in\mathbf{P}}T_p\Big)=(\mathbf{Q}/\mathbf{Z})^n\oplus(\bigoplus_{p\in\mathbf{P}}T_p)$$

according to Corollary 3.4.

3.6. Corollary. Let G be a periodic Abelian group which admits a totally minimal topology. Then

(9) $$G=(\mathbf{Q}/\mathbf{Z})^n\oplus(\bigoplus_{p\in\mathbf{P}}T_p),$$

where T_p is a compact p-group $(p\in\mathbf{P})$.

Indeed, G is precompact, by Theorem 3.8 in [4]. Then the completion $\hat{G}$ is an exotic torus, by Theorem 2.4, since $G\subset T(\hat{G})$, and consequently $T(\hat{G})$ is minimal and dense in $\hat{G}$. On the other hand, $G\supset T(\hat{G})$, since G is totally minimal, by Example 6 in [2]. Hence $G=T(\hat{G})$, and we can apply Corollary 3.5.

3.7. Proposition. A countable periodic Abelian group G admits a minimal topology if and only if there exist a non-negative integer n and for each $p\in\mathbf{P}$ — a finite p-group T_p such that for $H=(\mathbf{Q}/\mathbf{Z})^n\oplus(\bigoplus_{p\in\mathbf{P}}T_p)$ holds

(10) $$S(H)\subset G\subset H.$$

Proof. Let the group G admits a minimal topology. By Theorem 8 in [5], the group G is precompact. According to Theorem 3 from the same paper, for $X=G^*$ there is an exact sequence

$$0 \to A \to X \to \bigoplus_{p \in \mathbf{P}} F_p \to 0, \tag{11}$$

where A is a free subgroup of X of maximal rank, and F_p is a finite p-group ($p \in \mathbf{P}$). From (11) we obtain $T(X)=\bigoplus_{p \in \mathbf{P}} T_p$ where for each $p \in \mathbf{P}$ the group T_p is finite. Consider now the exact sequence

$$0 \to \bigoplus_{p \in \mathbf{P}} T_p \to X \to X' \to 0,$$

where X' is a strongly non-divisible torsion free group. Passing to the character groups, we receive the exact sequence

$$0 \leftarrow \prod_{p \in \mathbf{P}} T_p \leftarrow \hat{G} \leftarrow X'^* \leftarrow 0 \tag{12}$$

of compact Abelian groups and continuous homomorphisms. Since X'^* is the component of zero in $\hat{G}$, from the exactness of (12) it follows

$$T(\hat{G})=(\mathbf{Q}/\mathbf{Z})^n \oplus (\bigoplus_{p \in \mathbf{P}} T_p). \tag{13}$$

Clearly

$$G \subset T(\hat{G}). \tag{14}$$

On the other hand G is minimal and dense in $\hat{G}$. Then each non-zero closed subgroup of $\hat{G}$ intersects $G \setminus \{0\}$, by Theorem 2 in [6]. That is why $S(T(\hat{G})) \subset G$. Now (13) and (14) prove the necessity.

Conversely, let G satisfy (10). Take an arbitrary strongly non-divisible torsion free group X' with $r(X')=n$. Let X be a group for which the sequence

$$0 \to \bigoplus_{p \in \mathbf{P}} T_p \to X \to X' \to 0$$

is exact. Then X^* is exotic torus, and

$$T(X^*)=(\mathbf{Q}/\mathbf{Z})^n \oplus (\bigoplus_{p \in \mathbf{P}} T_p).$$

Now every isomorphism between H and $T(X^*)$ endows H with a minimal topology. Since each non-zero subgroup of H intersects $S(H) \setminus \{0\}$, and hence $G \setminus \{0\}$, the topology induced on G is minimal. **Q.E.D.**

REFERENCES

1. Чуканов, В.; Върху някои подкласове на абелевите категории. Год. Соф. унив., Мат. фак., 62 (1967/68), 325—241.
2. Dikranjan, D., Prodanov, Iv.: Totally minimal topological groups. Ann. de l'Univ. de Sofia, Fac. des Math., 69 (1974/75), 5—11.
3. Doitchinov, D.: Produits de groupes topologiques minimaux. Bull. Sc. math., 2ᵉ série, 96 (1972), 59—64.
4. Prodanov, Iv.: Some minimal group topologies are precompact. Math. Ann., 227 (1977), 117—125.
5. Prodanov, Iv.: Minimal topologies on countable Abelian groups Ann. de l'Univ. de Sofia, Fac. des Math., 70 (1975/76), 107—118.
6. Stephenson, R. M.: Minimal topological groups. Math. Ann., 192. (1971), 193—195.

Постъпила на 19.I.1977 г.

ЕДИН КЛАС ОТ КОМПАКТНИ АБЕЛЕВИ ГРУПИ

Д. Дикранян, И. Проданов

(РЕЗЮМЕ)

В работата се разглеждат компактните абелеви групи, в които всяка ненулева затворена подгрупа съдържа ненулеви периодични елементи. Поради приликата им със стандартните торуси тези групи са наречени екзотични торуси и са описани с помощта на дуалните им групи. Последните се характеризират с това, че факторгрупите им никога не са делими. Ето защо те са наречени силно неделими.

По-важните свойства на силно неделимите групи се съдържат в теореми 1.4, 1.9 и предложения 1.5 и 1.6. По-важните свойства на екзотичните торуси се съдържат в теореми 2.4* и 2.4. Оказва се, че те са тясно свързани с минималните топологични групи. Периодичната част на един екзотичен торус е минимална и това е характеристично свойство на класа на екзотичните торуси. С помощта на екзотични торуси е доказана теорема 3.3, която гласи, че една периодична делима група допуска предкомпактна минимална топология точно когато е изоморфна с $(\mathbf{Q}/\mathbf{Z})^n$ ($n=0, 1, 2, \dots$). Всички такива топологии са и тотално минимални и се получават от влагането на $(\mathbf{Q}/\mathbf{Z})^n$ в свързани екзотични торуси. Накрая е показано как могат да се опишат всички минимални топологии в изброима периодична абелева група.