L_2-CONVERGENCE OF RANDOM PROCESSES DEFINED BY STOCHASTIC EQUATIONS

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A wide class of random processes with a continuous time can be obtained as a solution of a stochastic differential equation (SDE). It is convenient to consider two types of SDE: W-SDE and P-SDE which include stochastic integrals about the Wiener process $w(t)$ and the Poisson random measure $p([0, t], A)$, respectively. There are many problems stated and solved for M-SDE — a more general class of SDE including stochastic integrals about a martingale.

In this paper we will consider W-stochastic ordinary differential equations with delay (W-SODED) the right side of which depend on a small parameter ε. Let the random process $\eta_\varepsilon(t)$ be a solution of such an equation. The aim is to find another, more simple, random process approximating $\eta_\varepsilon(t)$ when $\varepsilon \to 0$. A concrete scheme is given for transforming the coefficients of W-SODED and in result we obtain W-stochastic ordinary differential equation (W-SODE) which is already without delay and without dependence on ε. Let the random process $x(t)$ be its solution. One of the main results (theorem 1) is the following: for an arbitrary finite t $\eta_\varepsilon(t/\varepsilon) \xrightarrow{L_2} x(t)$ when $\varepsilon \to 0$.

Some results similar to the presented here were communicated by the author at the International Conference on Differential Equations, Russe, June 1975 (see [11]).

The results of the present paper extend to W-SODED some of the results of A. Halanay [2] and J. Hale [3] and also generalize the ones of V. Kolomiets [6].

1. PRELIMINARIES. MAIN RESULT

Let $(\Omega, F, \mathbf{P})$ be a complete probability space and $\{F_t\}$, $t \geqq 0$, be a family of nondecreasing σ-algebras, $F_s \subset F_t$ for $s<t$ and $F_t \subset F$, $t \geqq 0$. The symbol $\mathbf{E}$ denotes an integration with respect to the measure $\mathbf{P}$.

Let $w(t)$, $t \geqq 0$, be a standard Wiener process adapted with the family $\{F_t\}$. The stochastic integrals used below are in Ito's sense [1, 5].

The general form of the W-stochastic ordinary differential equation with delay (W-SODED) is the following:

$$\eta(t)=\begin{cases}\varphi(t), & -\Delta\leqq t<0,\\ \varphi(t)+\int_0^t a(s,\eta(s-\Delta),\eta(s))\,ds+\int_0^t b(s,\eta(s-\Delta),\eta(s))\,dw(s), & t\geqq 0.\end{cases}$$

Here $\varphi(t)$, $a(s, x, y)$ and $b(x, y, z)$ are given functions and the delay Δ is a constant.

In this paper we will study the family of random processes $\{\eta_\varepsilon(t), \varepsilon\in(0, \varepsilon_0], t\geqq 0\}$ where $\eta_\varepsilon(t)$ is defined by the equation

$$(1)\quad \eta_\varepsilon(t)=\begin{cases}x_\varepsilon, & -\Delta\leqq t<0,\\ x_\varepsilon+\varepsilon\int_0^t a(s,\eta_\varepsilon(s-\Delta),\eta_\varepsilon(s))\,ds+\varepsilon^\alpha\int_0^t b(s,\eta_\varepsilon(s-\Delta),\eta_\varepsilon(s))\,dw(s) & \\ & t\geqq 0.\end{cases}$$

The equation (1) is said to be W-SODED in standard form. There again $\Delta=\text{const}>0$, $\alpha\geqq 1/2$ and x_ε is a random variable not depending on the process $w(t)$, $\mathrm{E}\{x_\varepsilon^2\}<\infty$ for each ε.

We shall assume that the following conditions are fulfilled:

(A_1) The functions $a(s, x, y)$ and $b(s, x, y)$ are: measurable functions of their arguments; continuous in s; satisfy the local Lipschitz condition in x, y, i. e. $|a(s, x, y)-a(s, x', y')|+|b(s, x, y)-b(s, x', y')\leqq C_N(|x-x'|+|y+y'|)$ for $|x|, |y|, |x'|, |y'|\leqq C_N$, $C_N=\text{const}>0$; bounded by a linear function, i. e. $|a(s, x, y)|+|b(s, x, y)|\leqq K(1+|x|+|y|)$, $K=\text{const}>0$.

(A_2) There is a $\{F_t\}$-adapted Wiener process $w_0(t)$, $t\geqq 0$, such that for every finite t $\sqrt{\varepsilon}\,w(t/\varepsilon)\xrightarrow{L_2} w_0(t)$ at $\varepsilon\to 0$.

It follows from the results of [1, 5] that under condition (A_1) the W-SODED (1) has a unique solution $\eta_\varepsilon(t)$, $t\geqq 0$.

Let us consider the new functions $a_1(s, x)$ and $b_1(s, x)$ where

$$a_1(s, x)=a(s, x, x),\quad b_1(s, x)=b(s, x, x).$$

Our basic assumption is the following:

(A_3) There are functions $a_2(x)$ and $b_2(x)$ such that

$$\lim_{T\to\infty}(1/T)\int_0^T[a_1(s, x)-a_2(x)]\,ds=0,$$

$$\lim_{T\to\infty}(1/T)\int_0^T[b_1(s, x)-b_2(x)]^2\,ds=0$$

uniformly in x.

By aid of the functions $a_2(x)$ and $b_2(x)$, the Wiener process $w_0(t)$ and some new random variable x_0 not depending on $w_0(t)$ we construct the following W-SODE (without delay and without dependence on ε):

(2) $$x(t)=x_0+\int_0^t a_2(x(s))\,ds+\int_0^t b_2(x(s))\,dW_0(s).$$

It is not difficult to see that $a_2(x)$ and $b_2(x)$ satisfy conditions similar (in fact more simple) to those for $a(\cdot)$ and $b(\cdot)$. Therefore the W-SODE (2) has a unique solution $x(t)$, $t\geqq 0$.

Both processes, $\eta_\varepsilon(t)$ and $x(t)$, are continuous with probability 1 but $x(t)$ is simpler. It is a homogeneous Markov process while $\eta_\varepsilon(t)$ is non-homogeneous and non-Markov.

We are interested in the following question. Is there any connection between the random processes $\eta_\varepsilon(t)$ from W-SODED (1) and $x(t)$ from W-SODE (2).

The answer of this question is contained in theorem 1.

Theorem 1. Let the conditions (A_1), (A_2) and (A_3) be fulfilled and $\alpha=1/2$. If $\delta(\varepsilon)=E\{(x_\varepsilon-x_0)^2\}\to 0$ at $\varepsilon\to 0$ then for each finite t $\eta_\varepsilon(t/\varepsilon)\xrightarrow{L_2} x(t)$ at $\varepsilon\to 0$, i. e. for an arbitrary T_1, $0<T_1<\infty$,

$$\lim_{\varepsilon\to 0}\sup_{0\leqq t\leqq T_1} E\{(\eta_\varepsilon(t/\varepsilon)-x(t))^2\}=0.$$

2. SUPPLEMENTARY RESULTS

Now we shall formulate and prove a few other results which will be used for proving the theorem 1.

By the aid of the functions $a_1(s, x)$, $b_1(s, x)$ and $a_2(x)$, $b_2(x)$ we obtain two W-SODE in standard form:

(3) $$\zeta_\varepsilon(t)=x_\varepsilon+\varepsilon\int_0^t a_1(s, \zeta_\varepsilon(s))\,ds+\varepsilon^\alpha\int_0^t b_1(s, \zeta_\varepsilon(s))\,dw(s),$$

(4) $$v_\varepsilon(t)=x_\varepsilon+\varepsilon\int_0^t a_2(v_\varepsilon(s))\,ds+\varepsilon^\alpha\int_0^t b_2(v_\varepsilon(s))\,dw(s).$$

Theorem 2. Let the condition (A_1) be fulfilled and $\alpha\geqq 1/2$. Then for an arbitrary T_2, $0<T_2<\infty$,

$$\lim_{\varepsilon\to 0}\sup_{0\leqq t\leqq T_2\varepsilon^{-1}} E\{(\eta_\varepsilon(t)-\zeta_\varepsilon(t))^2\}=0.$$

Theorem 3. Under the conditions (A_1), (A_3) and $\alpha\geqq 1/2$ for an arbitrary T_3, $0<T_3<\infty$,

$$\lim_{\varepsilon\to 0}\sup_{0\leqq t\leqq T_3\varepsilon^{-1}} E\{(\zeta_\varepsilon(t)-v_\varepsilon(t))^2\}=0.$$

Theorem 4. If the conditions (A_1), (A_2), (A_3) are fulfilled, $\alpha=1/2$ and $\delta(\varepsilon)\to 0$ at $\varepsilon\to 0$ (as in theorem 1) then for an arbitrary T_4, $0<T_4<\infty$,

$$\lim_{\varepsilon\to 0}\sup_{0\leqq t\leqq T_4} E\{(v_\varepsilon(t/\varepsilon)-x(t))^2\}=0.$$

3. PROOFS OF THEOREMS 1—4

It is more convenient when we prove these theorems to observe that (A_1) contains the uniform Lipschitz condition with some constant K_0. The proofs in the case of a local Lipschitz condition can be obtained after a passage to the limit (see [5], p. 45).

Proof of theorem 2. Let us take some fixed $T_2>0$, $\alpha=1/2$ and t from the interval $[0, T_2\varepsilon^{-1}]$.

If we put

$$I_1(s)=a(s, \eta_\varepsilon(s-\Delta), \eta_\varepsilon(s))-a_1(s, \zeta_\varepsilon(s)),$$

$$I_2(s)=b(s, \eta_\varepsilon(s-\Delta), \eta_\varepsilon(s))-b_1(s, \zeta_\varepsilon(s)),$$

then for the difference $\eta_\varepsilon(t)-\zeta_\varepsilon(t)$ we get

$$(\eta_\varepsilon(t)-\zeta_\varepsilon(t))^2\leqq 2\left(\varepsilon\int_0^t I_1(s)\,ds\right)^2+2\left(\sqrt{\varepsilon}\int_0^t I_2(s)\,dw(s)\right)^2.$$

Using the Hölder inequality and the properties of the stochastic integrals [5] we find

$$r(\varepsilon, t)=\mathbf{E}\{(\eta_\varepsilon(t)-\zeta_\varepsilon(t))^2\}\leqq 2\varepsilon^2 t\int_0^t \mathbf{E}\{I_1^2(s)\}\,ds+2\varepsilon\int_0^t \mathbf{E}\{I_2^2(s)\}\,ds.$$

It is easy to see that (A_1) gives (further on, $K_i=\text{const}$)

$$\int_0^t \mathbf{E}\{I_1^2(s)\}\,ds\leqq 2K_1\int_0^t r(\varepsilon, s)\,ds+2K_1\int_0^t \mathbf{E}\{(\eta_\varepsilon(s-\Delta)-\zeta_\varepsilon(s))^2\}\,ds.$$

According to the results of [5, p. 48] and theorem 2 in [10] for each $t\in[0, T_2\varepsilon^{-1}]$ we have $\mathbf{E}\{(\eta_\varepsilon(s-\Delta)-\eta_\varepsilon(s))^2\}\leqq K_2\varepsilon\Delta$, where K_2 is a constant not depending on ε, Δ and s. Let us note that in the case $\varepsilon=1$ similar estimate can be concluded from the results in [9]. Therefore

$$\int_0^t \mathbf{E}\{(\eta_\varepsilon(s-\Delta)-\zeta_\varepsilon(s))^2\}\,ds\leqq 2\int_0^t \mathbf{E}\{(\eta_\varepsilon(s-\Delta)-\eta_\varepsilon(s))^2\}\,ds$$

$$+2\int_0^t \mathbf{E}\{(\eta_\varepsilon(s)-\zeta_\varepsilon(s))^2\}\,ds\leqq 2K_2 t\varepsilon\Delta+2\int_0^t r(\varepsilon, s)\,ds.$$

Thus

$$\int_0^t \mathbf{E}\{I_1^2(s)\}\,ds\leqq K_3\int_0^t r(\varepsilon, s)\,ds+K_4 t\varepsilon\Delta,$$

where the constants K_3 and K_4 do not depend on ε, Δ and t.

Evidently, a similar estimate will also be valid for $\int_0^t E\{I_2^2(s)\}\, ds$.

Therefore

$$r(\varepsilon, t) \leq 2\varepsilon^2 t \left(K_3 \int_0^t r(\varepsilon, s)\, ds + K_4 t \varepsilon \Delta\right) + 2\varepsilon \left(K_5 \int_0^t r(\varepsilon, s)\, ds + K_6 t \varepsilon \Delta\right).$$

Since $t \in [0, T_2 \varepsilon^{-1}]$ then $\varepsilon t \leq T_2$ and

$$r(\varepsilon, t) \leq \varepsilon K_7 \int_0^t r(\varepsilon, s)\, ds + K_8\, \varepsilon \Delta,$$

where K_7 and K_8 again do not depend on ε, Δ and t. Applying the well-known Gronwall-Bellman inequality we obtain

$$r(\varepsilon, t) \leq K_8\, \varepsilon \Delta \exp[K_7\, \varepsilon t] \leq K_8\, \varepsilon \Delta \exp[K_7 T_2].$$

It means that $\lim\limits_{\varepsilon \to 0} r(\varepsilon, t) = 0$ for $t \in [0, T_2 \varepsilon^{-1}]$.

Theorem 2 is proved.

Proof of theorem 3. It suffice to take $\alpha = 1/2$. From (3) and (4) we have

$$\zeta_\varepsilon(t) - v_\varepsilon(t) = \varepsilon \int_0^t J_1(s)\, ds + \sqrt{\varepsilon} \int_0^t J_2(s)\, dw(s),$$

where

$$J_1(s) = J_{11}(s) + J_{12}(s), \quad J_2(s) = J_{21}(s) + J_{22}(s),$$

$$J_{11}(s) = a_1(s, \zeta_\varepsilon(s)) - a_1(s, v_\varepsilon(s)), \quad J_{12}(s) = a_1(s, v_\varepsilon(s)) - a_2(v_\varepsilon(s)),$$

$$J_{21}(s) = b_1(s, \zeta_\varepsilon(s)) - b_1(s, v_\varepsilon(s)), \quad J_{22}(s) = b_1(s, v_\varepsilon(s)) - b_2(v_\varepsilon(s)).$$

Further on, if $\rho(\varepsilon, t) = E\{(\zeta_\varepsilon(t) - v_\varepsilon(t))^2\}$ then

$$\rho(\varepsilon, t) \leq 4\varepsilon^2 t \int_0^t E\{J_{11}^2(s)\}\, ds + 4\varepsilon^2 E\left\{\left(\int_0^t J_{12}(s)\, ds\right)^2\right\}$$

$$+ 4\varepsilon \int_0^t E\{J_{21}^2(s)\}\, ds + 4\varepsilon \int_0^t E\{J_{22}^2(s)\}\, ds.$$

From the Lipschitz condition for $a_1(\cdot)$ and $b_1(\cdot)$ follows that

$$\int_0^t E\{J_{11}^2(s)\}\, ds + \int_0^t E\{J_{21}^2(s)\}\, ds \leq K_9 \int_0^t \rho(\varepsilon, s)\, ds.$$

Using the method of proving of theorem 1 in [4] and theorem 2.1 in [8] we come to the following relations: for $t \in [0, T_3 \varepsilon^{-1}]$ and $\varepsilon \to 0$

$$\delta_1(\varepsilon) = 4\varepsilon^2 \mathbf{E}\left\{\left(\int_0^t J_{12}(s)ds\right)^2\right\} \to 0, \quad \delta_2(\varepsilon) = 4\varepsilon \int_0^t \mathbf{E}\{J_{22}^2(s)\}\, ds \to 0.$$

After applying the Gronwall-Bellman inequality we get

$$\sup_{0 \leq t \leq T_3 \varepsilon^{-1}} \rho(\varepsilon, t) \leq K_{10}(\delta_1(\varepsilon) + \delta_2(\varepsilon)) \to 0 \text{ when } \varepsilon \to 0.$$

This completes the proof of theorem 3.

Proof of theorem 4. The proof follows from the concrete structures of the processes $v_\varepsilon(t)$ and $x(t)$ and uses essentially the condition (A_2) and the convergence properties of the stochastic integrals with respect to the Wiener process.

Proof of theorem 1. Firstly we rewrite the statements of theorems 2 and 3 in another form:

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T_2} \mathbf{E}\{(\eta_\varepsilon(t/\varepsilon) - \zeta_\varepsilon(t/\varepsilon))^2\} = 0$$

and

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T_3} \mathbf{E}\{(\zeta_\varepsilon(t/\varepsilon) - v_\varepsilon(t/\varepsilon))^2\} = 0.$$

From these relations and theorem 4, since T_1, T_2, T_3 and T_4 are arbitrary constants, we get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T_1} \mathbf{E}\{(\eta_\varepsilon(t/\varepsilon) - x(t))^2\} \\
&\leq \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T_1} \mathbf{E}\{(\eta_\varepsilon(t/\varepsilon) - \zeta_\varepsilon(t/\varepsilon))^2\} \\
&+ \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T_1} \mathbf{E}\{(\zeta_\varepsilon(t/\varepsilon) - v_\varepsilon(t/\varepsilon))^2\} \\
&+ \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T_1} \mathbf{E}\{(v_\varepsilon(t/\varepsilon) - x(t))^2\} = 0.
\end{aligned}$$

Theorem 1 is proved.

4. SOME COMMENTS

4.1. The results proved in this paper are valid also in the case when the coefficients $a(\cdot)$ and $b(\cdot)$ are random functions. More precisely: let $a(\cdot) = a(s, x, y, \omega)$, $b(\cdot) = b(s, x, y, \omega)$, $s \geq 0$, $x, y \in R_1$, $\omega \in \Omega$. In this case we must assume that they satisfy (A_1) with probability 1 and are $\{F_s\}$-adapted for an arbitrary pair (x, y). Let $a_1(s, x, \omega) = a(s, x, x, \omega)$, $b_1(s, x, \omega) = b(s, x, x, \omega)$. Instead of the condition (A_3) we require fulfilling the strong law of the large numbers, i. e. there are random functions $a_2(x, \omega)$ and $b_2(x, \omega)$ such that

$$\lim_{T\to\infty}(1/T)\mathrm{E}\left\{\int_0^T [a_1(s,\ x,\ \omega)-a_2(x,\ \omega)]\,ds\right\}=0,$$

$$\lim_{T\to\infty}(1/T)\mathrm{E}\left\{\int_0^T [b_1(s,\ x,\ \omega)-b_2(x,\ \omega)]^2\,ds\right\}=0.$$

Since (when $a(\cdot)$ and $b(\cdot)$ are random) the proof of the statements analogical to those in theorems 1—4 does not require to develop new ideas we do not give any details.

4.2. Let us recall that in theorem 1 $\alpha=1/2$ and in theorems 2—4 $\alpha\geqq 1/2$. It turned out that the case $\alpha>1/2$ is also interesting. We will formulate the following result.

Theorem 5. Let the functions $a(s,\ x,\ y)$ and $b(s,\ x,\ y)$ (from part 1) satisfy (A_1), (A_2) be fulfilled only for $a(\cdot)$, $\alpha>1/2$ and $T_5>0$ be an arbitrary finite constant. If $\mathrm{E}\{|x_\varepsilon|^k\}<\infty$, $x_\varepsilon\xrightarrow{L_k}x_0$ for some constant x_0 at $\varepsilon\to 0$, $k>0$ then for each $r\in(0,\ k]$

$$\eta_\varepsilon(t/\varepsilon)\xrightarrow{L_r}x(t),$$

where $\eta_\varepsilon(t)$ is defined by W-SODED (1) and $x(t)$ is a deterministic function satisfying the equation

$$x(t)=x_0+\int_0^t a_2(x(s))\,ds.$$

The proof of this theorem follows from the above reasonings (parts 1—3) and uses the results from [7].

4.3. If the coefficients $a(\cdot)$ and $b(\cdot)$ are nonrandom (as in parts 1—3) then the process $\eta_\varepsilon(t)$ is a non-Markov process and at the same time each of the processes $\zeta_\varepsilon(t)$, $v_\varepsilon(t)$ and $x(t)$ is a Markov one. Therefore, theorems 1—5 offer a possibility, in principle, for approximating the non-Markov process $\eta_\varepsilon(t)$ by another process (some of $\zeta_\varepsilon(t)$, $v_\varepsilon(t)$ and $x(t)$) which is already a Markov one.

4.4. It seems to us that the theory of stochastic equations with delay can be used successfully for treating many problems of mechanics and engineering. It is enough to recall that the behaviour of many practical systems with aftereffects is described namely by differential equations with delay. The W-stochastic differential equations with delay are their natural generalizations.

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L_2-СХОДИМОСТ НА СЛУЧАЙНИ ПРОЦЕСИ, ДЕФИНИРАНИ СЪС СТОХАСТИЧНИ УРАВНЕНИЯ

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Нека случайният процес $\eta_\varepsilon(t)$, $t \geqq 0$, е решение на W-СОДУЗ (W-стохастично обикновено диференциално уравнение със закъснение) и с дясна част, зависеща от параметъра ε.

Целта е да се намери друг, по-прост случаен процес, апроксимиращ $\eta_\varepsilon(t)$ при малки стойности на ε. Предложена е схема за преобразуване на коефициентите на W-СОДУЗ, в резултат на което получаваме W-СОДУ — без закъснение и без зависимост от ε. Нека $x(t)$ е неговото решение.

Основният резултат на работата е следният: за произволно крайно t имаме $\eta_\varepsilon(t/\varepsilon) \xrightarrow{L_2} x(t)$ при $\varepsilon \to 0$.

Доказаните тук резултати пренасят за W-СОДУЗ някои от резултатите на Халанай [2] и Хейл [3] и обобщават тези на Коломиец [6].