A PROPERTY OF PRECOMPACT MINIMAL ABELIAN GROUPS

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Introduction

A Hausdorff topological group G is said to be minimal if every continuous group isomorphism $h:G\to G_1$, where G_1 is a Hausdorff topological group, is homeomorphism. Obviously, every compact topological group is minimal. Doïtchinov [2] gives examples of non-compact minimal topological groups.

A Hausdorff topological group G is said to be totally minimal if every continuous group epimorphism $h:G\to G_1$, where G_1 is a Hausdorff topological group, is open. Clearly, every compact group is totally minimal and every totally minimal group is minimal.

The main purpose of this paper is to establish some cardinal invariants of compact Abelian groups and to prove that every precompact minimal Abelian group may be topologically imbedded in a precompact totally minimal Abelian group with the same cardinality (see sec. 2). The main technical tool which is used in sec. 2 is Theorem 2 (see sec. 1). As a whole the paper is a continuation of a part of [3], and uses the methods developed there. The author wishes to thank I. Prodanov for the stating of the problem and for the permanent encouragement.

We use the following notations: P — the set of all primes; $\mathbf{Z}$ — the aditive group of the integers; $\mathbf{Z}(p^\infty)$ — the quotient group of the rationals with denominators powers of p, over $\mathbf{Z}$ $(p\in P)$; $S(X)$ — the socle of the p-group X (the set of all elements of X with period p); $A^{(\tau)}=\bigoplus_{\tau} A$, where A is a group and τ is a cardinal; X^* — the group of all continuous characters of the topological group X; $\mathbf{Z}_p$ — the compact group of p-adic numbers. It is known that $\mathbf{Z}_p^*=\mathbf{Z}(p^\infty)$; $\mathbf{T}^n=\mathbf{R}^n/\mathbf{Z}^n$, where $\mathbf{R}$ is the topological aditive group of the reals; $\mathfrak{a}_0=\operatorname{card}\mathbf{Z}$, $\mathfrak{c}=\operatorname{card}\mathbf{R}$.

Let G be a locally compact Abelian group and A be a subgroup of G. By $A^\perp$ we denote the group of all continuous characters χ of G such that $\chi(A)=\{0\}$. If B is a subgroup of G^*, using the Pontrjagin's duality, we shall identify $B^\perp$ with the group of all elements x of G such that $\chi(x)=0$ for every $\chi\in B$. It is well known that for every family $\{A_\alpha\}_\alpha$ of subgroups of G we have $\left(\sum_\alpha A_\alpha\right)^\perp=\bigcap_\alpha A_\alpha^\perp$, where by $\sum_\alpha A_\alpha$ is denoted the minimal subgroup of G which contains A_α for every α.

A subgroup G' of the topological group G is said to be totally dense in G if for every closed normal subgroup H of G the group $H \cap G'$ is dense in H.

We make use of the following proposition which is proved in [1].

Proposition 1. Let G be a Hausdorff topological group and G' be a dense subgroup of G. Then G' is totally minimal iff G is totally minimal and G' is totally dense in G.

1. THE MAIN STATEMENT

Theorem 1. Let τ_0 be an infinite cardinal and τ be the minimal cardinal with the property $\max(\mathfrak{c}, 2^{\tau}) > \tau_0$. Let X be an Abelian group. Then the following conditions are equivalent:

i) X is the group of the continuous characters of a precompact totally minimal Abelian group, with cardinality at most τ_0;

ii) X is the group of the continuous characters of a precompact totally minimal Abelian group, with cardinality, less than $\max(\mathfrak{c}, 2^{\tau})$;

iii) X is the group of the continuous characters of a precompact minimal Abelian group, with cardinality at most τ_0;

iv) X is the group of the continuous characters of a precompact minimal Abelian group, with cardinality, less than $\max(\mathfrak{c}, 2^{\tau})$;

v) the set of maximal proper subgroups of X has a cardinality at most τ_0, and for each prime p there does not exist an epimorphism $X \to \mathbf{Z}(p^{\infty})^{(\tau+2)}$;

vi) the set of maximal proper subgroups of X has a cardinality, less than $\max(\mathfrak{c}, 2^{\tau})$, and for each prime p there does not exist an epimorphism $X \to \mathbf{Z}(p^{\infty})^{(\tau+2)}$;

vii) the rank of the group X is less than $\max(\mathfrak{a}_0, \tau)$ and if A is a free subgroup of X such that X/A is periodic, then

$$X/A = \bigoplus_{p \in \mathbf{P}} (\mathbf{Z}(p^{\infty})^{(\tau_p)} \oplus F_p),$$

where $\tau_p < \tau + 2$, F_p denotes a suitable p-group without non-zero divisible subroups and card $F_p < \max(\mathfrak{a}_0, \tau)$ $(p \in \mathbf{P})$.

Proof. We prove the theorem using the following diagram:

$$\begin{tikzcd}
& \text{ii)} \arrow[r, Rightarrow] & \text{iv)} \arrow[dr, Rightarrow] & & & \\
\text{i)} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \text{vi)} \arrow[r, Rightarrow] & \text{vii)} \arrow[r, Rightarrow] & \text{i).} \\
& \text{iii)} \arrow[r, Rightarrow] & \text{v)} \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

The implications i) $\Rightarrow$ ii), i) $\Rightarrow$ iii), ii) $\Rightarrow$ iv), v) $\Rightarrow$ vi) are obvious and the proofs of the implications iii) $\Rightarrow$ v) and iv) $\Rightarrow$ vi) are analogical to the corresponding implications of Theorem 3 in [3].

Proof of vi) $\Rightarrow$ vii). We consider the case $\tau > \mathfrak{a}_0$, because the case $\tau \leq \mathfrak{a}_0$ is considered in [3].

Let X be an Abelian group, satisfying the condition vi). Then for $p \in \mathbf{P}$ we have card $\mathbf{Z}(p^{\infty})^{(\tau)} = \tau$ and there does not exist an epimorphism $S \to \mathbf{Z}$

$(p^\infty)^{(\tau)}$. Hence the rank of X is less than τ. Let A be a free subgroup of X such that X/A is periodic. Then $X/A = \bigoplus_{p \in P} T_p$, where T_p is a p-group $(p \in \mathbf{P})$.

The main difficulty is to prove that card $T_p < \tau$ for every prime p. Let us fix a prime p. For every positive integer k there exist epimorphisms

$$X \to T_p \to p^k T_p \to p^k T_p/p^{k+1} T_p.$$

On the other hand $p^k T_p/p^{k+1} T_p$ is a vector space over $\mathbf{Z}/p\mathbf{Z}$ and if we assume card $(p^k T_p/p^{k+1} T_p) \geqq \tau$ there would be at least 2^τ hyperplanes in $p^k T_p/p^{k+1} T_p$, and hence there would be at least 2^τ maximal proper subgroups in X, which contradicts vi). Hence card $(p^k T_p/p^{k+1} T_p) < \tau$ $(k=0, 1, 2, \ldots)$.

Let us assume now that card $T_p \geqq \tau$. It follows from the exactness of the sequences

$$0 \to p^{k+1} T_p \to p^k T_p \to p^k T_p/p^{k+1} T_p \to 0$$

that card $(p^k T_p) =$ card T_p $(k=1, 2, \ldots)$.

But if T is a non-countable p-group, then card $T =$ card $S(T)$. Indeed, if for every positive integer n T_n denotes the group of all elements $x \in T$ such that $p^n x = 0$, then

$$S(T) = T_1 \subset T_2 \subset \cdots \subset T_n \subset T_{n+1} \subset \cdots \text{ and } T = \bigcup_{n=}^{\infty} T_n.$$

For the homomorphism $\varphi\colon T_{n+1} \to T_n$, defined by $\varphi(x) = px$, we have Ker $\varphi \subset T_1$ and hence card $T_{n+1} \leqq$ (card T_n) (card T_1). This implies card $T_n \leqq$ (card $T_1)^n$ for every $n=1, 2, \ldots$ Now we obtain that T_1 is infinite and hence card $T_n =$ card T_1 $(n=1, 2, \ldots)$. Now card $T =$ card $S(T)$ follows immediately.

Let us consider the sequence $Y_0 \supset Y_1 \supset \cdots \supset Y_k \supset \cdots$, where $Y_k = S(p^k T_p)$. Then card $Y_k =$ card T_p $(k=0, 1, \ldots)$. On the other hand

$$\text{card } (Y_k/Y_{k+1}) \leqq \text{card } (p^k T_p/p^{k+1} T_p) < \tau$$

and hence card $(Y_0/Y_k) < \tau$ $(k=1, 2, \ldots)$. Since Y_0 is an infinite dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$, we are able to find a sequence $Z_1, Z_2, \ldots, Z_n, \ldots$ of vector subspaces of Y_0 such that $Y_0 = \bigoplus_{n=1}^{\infty} Z_n$ and card $Z_n =$ card Y_0 $(n=1, 2, \ldots)$. For every positive integer n there exists a monomorhism $Z_n/Z_n \cap Y_n \to Y_0/Y_n$. Therefore, card $Z_n/Z_n \cap Y_n \leqq$ card $Y_0/Y_n < \tau \leqq$ card Z_n. Hence card $Z_n/Z_n \cap Y_n =$ card Z_n. Let us denote by B_n a Hamel's basis of $Z_n \cap Y_n$. Then card $B_n =$ card $Z_n \cap Y_n \geqq \tau$, and hence there exists a surjection $\lambda_n\colon B_n \to S(\mathbf{Z}(p^\infty))^{(\tau)}$. Obviously, λ_n can be extended to an epimorhism $\varphi_n\colon Z_n \cap Y_n \to S(\mathbf{Z}(p^\infty))^{(\tau)}$. Now we obtain the homomorphism

$$\varphi = \bigoplus_{n=1}^{\infty} \varphi_n \colon \bigoplus_{n=1}^{\infty} (Z_n \cap Y_n) \to \mathbf{Z}(p^\infty)^{(\tau)}.$$

Since $\mathbf{Z}(p^\infty)^{(\tau)}$ is a divisible group, φ can be extended to a homomorphism $\psi\colon T \to \mathbf{Z}(p^\infty)^{(\tau)}$. It is easy to see that $\psi(p^k T_p) \supset S(\mathbf{Z}(p^\infty)^{(\tau)})$ for every

$k=0, 1, \ldots$, which implies that ψ is an epimorphism. Thus we found an epimorphism $X \to \mathbf{Z}(p^\infty)^{(\tau)}$ which contradicts vi). Hence card $T_p<\tau$ for every prime p.

Let D_p be the maximal divisible subgroup of T_p. Then $T_p=D_p\oplus F_p$, where F_p has no non-zero divisible subgroups. It follows from the structural theorem for divisible groups that $D_p=\mathbf{Z}(p^\infty)^{(\tau_p)}$. From card $T_p<\tau$ it follows $\tau_p<\tau$ and card $F_p<\tau$. In such a way the implication iv)$\Rightarrow$vii) is proved.

Proof of vii)$\Rightarrow$i). Assume that X is an Abelian group, satisfying the condition vii) and A is a free subgroup of X such that X/A is periodic. Consider the decomposition of X/A into a sum of p-groups: $X/A=\bigoplus_{p\in P} T_p$.

Now we prove that for every prime p there exists a totally dense subgroup of T_p^* with cardinality at most τ_0. There are two cases.

Case A): $\tau\leqq\aleph_0$. Then $\tau=0$ and hence T_p is finite or $T_p=\mathbf{Z}(p^\infty)\oplus F_p$, where F_p is finite. The first subcase is trivial and we consider $T_p=\mathbf{Z}(p^\infty)\oplus F_p$. It is easy to see that the set of all closed subgroups of T_p^* is countable. Now we are able to find a countable totally dense subgroup of T_p^*. In fact, it may be proved that $\mathbf{Z}\times F_p$ is totally dense in $\mathbf{Z}_p\times F_p$.

Case B): $\tau>\aleph_0$. Then card $T_p<\tau$, which implies $2^{\text{card } T_p}\leqq\tau_0$. Since the weight of the space T_p^* is equal to card T_p, we obtain card $T_p^*\leqq 2^{\text{card } T_p}\leqq\tau_0$.

There exists a totally dense subgroup N of $(X/A)^*$ with a caradinality at most τ_0. Indeed, $(X/A)^*=\prod_{p\in P} T_p^*$ and for every prime p there exists a totally dense subgroup N_p of T_p^* such that card $N_p\leqq\tau_0$. Then if $N=\bigoplus_{p\in P} N_p$, we have card $N\leqq\tau_0$. Let H be a closed subgroup of $(X/A)^*$. Then $H^\perp=\bigoplus_{p\in P}\Gamma_p$, where Γ_p is a subgroup of T_p, and hence $H=\bigcap_{p\in P}\Gamma_p^\perp$. Since $\Gamma_p^\perp\supset\prod_{q\neq p} T_q^*$ we find $\Gamma_p^\perp=\prod_{q\in P} H_q^p$, where H_q^p is a closed subgroup of T_q^*. Then $H=\prod_{q\in P} H_q$, where $H_q=\bigcap_{p\in P} H_q^p$, and hence $H\cap N$ is dense in H, which means that N is totally dense in $(X/A)^*$.

Now we prove that if Γ is a subgroup of A, then there exists a subgroup L_Γ of $(X/\Gamma)^*$ such that card $L_\Gamma\leqq\tau_0$ and for every closed subgroup H of $(X/\Gamma)^*$, satisfying $H^\perp\cap A/\Gamma=0$, the intersection $H\cap L_\Gamma$ is dense in H.

It follows from Zorn's lemma that there exists a subgroup B of A such that $B=\Gamma\oplus\Gamma'$ and A/B is periodic. Hence B is free and X/B is periodic. From the exactness of the sequence

$$0\to\Gamma\oplus\Gamma'\to X\to X/B\to 0,$$

it follows that the sequence

$$0\to\Gamma'\xrightarrow{i} X/\Gamma\xrightarrow{j} X/B\to 0$$

is also exact. By passing to the conjugate groups and homomorphisms we find the exact sequence

$$0 \leftarrow \Gamma'^{*} \xleftarrow{i_*} (X/\Gamma)^{*} \xleftarrow{j_*} (X/B)^{*} \leftarrow 0,$$

where $i_*(\chi)=\chi_0 i$, $j_*(\chi)=\chi_0 j$ are continuous homomorphisms between the corresponding compact groups.

Let T_Γ denotes the group of all periodic elements of X/Γ. It follows from $T_\Gamma \cap i(\Gamma')=0$ that $T_\Gamma^\perp+(i(\Gamma'))^\perp=(X/\Gamma)^*$. Hence $i_*(T_\Gamma^\perp)=\Gamma'^*$. The weight of the space Γ'^* is equal to card $\Gamma' \leqq$ card $A \leqq \tau_0$. Then there exists a subgroup M_Γ of $T_\Gamma^\perp$ such that $i_*(M_\Gamma)$ is dense in Γ'^* and card $M_\Gamma \leqq \tau_0$. On the other hand it follows from vii) that there exists a totally dense subgroup of $(X/B)^*$ with a cardinality at most τ_0. Consequently, there exists a subgroup N_Γ of $(X/\Gamma)^*$ such that card $N_\Gamma \leqq \tau_0$ and $j_*^{-1}(N_\Gamma)$ is totally dense in $(X/B)^*$. Let $L_\Gamma=M_\Gamma+N_\Gamma$. Clearly, card $L_\Gamma \leqq \tau_0$. Let H be a closed subgroup of $(X/\Gamma)^*$, satisfying $H^\perp \cap A/\Gamma=0$.Then $H^\perp \cap B/\Gamma=0$, which means that $H^\perp \cap i(\Gamma')=0$. Hence $H^\perp \subset T_\Gamma$ and $H \supset T_\Gamma^\perp$. Let x be an element of H. There exists a net $\{m_\alpha\}$ of elements of M_Γ such that $i_*(m_\alpha) \xrightarrow[\alpha]{} i_*(x)$. Since H is compact, without oss of generality we may assume that $m_\alpha \xrightarrow[\alpha]{} y \in H$. Then $i_*(x)=i_*(y)$ and hence $x-y=j_*(t)$. On the other hand $\tilde{H}=j_*^{-1}(H)$ is a closed subgroup of $(X/B)^*$ and since $j_*^{-1}(N_\Gamma)$ is totally dense in $(X/B)^*$, we find a net $n_\beta \xrightarrow[\beta]{} t$ such that $n_\beta \in \tilde{H} \cap j_*^{-1}(N_\Gamma)$. Then $j_*(n_\beta) \in H \cap N_\Gamma$ and $j_*(n_\beta) \xrightarrow[\beta]{} j_*(t)$. Therefore $m_\alpha+j_*(n_\beta) \xrightarrow[(\alpha,\beta)]{} x$. Since $m_\alpha+j_*(n_\beta) \in M_\Gamma+H\cap N_\Gamma$ and $M_\Gamma \subset T_\Gamma^\perp \subset H$ we find $m_\alpha+j_*(n_\beta) \in H \cap L_\Gamma$. Hence $H \cap L_\Gamma$ is dense in H. In this way the existence of L_Γ is proved.

Now we prove that if Γ is a subgroup of A, then there exists a subgroup G_Γ of X^* such that card $G_\Gamma \leqq \tau_0$ and if H is a closed subgroup of X^*, satisfying $H^\perp \cap A=\Gamma$, the intersection $H \cap G_\Gamma$ is dense in H.

Let φ be the canonical epimorphism $X \to X/\Gamma$. We have the exact sequence $X^* \xleftarrow{\varphi_*} (X/\Gamma)^* \leftarrow 0$. Now we consider the group $G_\Gamma=\varphi_*(L_\Gamma)$. Obviously, card $G_\Gamma \leqq \tau_0$. Let H be a closed subgroup of X^* such that $H^\perp \cap A=\Gamma$. For $\tilde{H}=\varphi_*^{-1}(H)$ we have $(\tilde{H})^\perp_{(X/\Gamma)^*}=\varphi((H)^\perp_{X^*})$. Then $(\tilde{H})^\perp_{(X/\Gamma)^*} \cap A/\Gamma=0$ and hence $\tilde{H} \cap L_\Gamma$ is dense in $\tilde{H}$. Therefore $H \cap G_\Gamma$ is dense in H. The existence of G_Γ is proved.

Let $G'=\sum_\Gamma G_\Gamma$, where Γ runs over the subgroups of A. Since there are at most τ_0 subgroups in A, we find card $G' \leqq \tau_0$. It is easy to see that G' is totally dense in X^* and from Proposition 1 we obtain that G' is totally minimal. Theorem 2 is proved.

2. SOME APPLICATIONS

Using the Pontrjagin's duality we obtain from Theorem 2:

Proposition 3. Let τ_0 be an infinite cardinal and τ be the minimal cardinal with the property $\max(\mathbf{c}, 2^\tau)>\tau_0$. Let G be a compact Abelian group. Then the following conditions are equivalent:

i) there exists a dense subgroup of G with cardinality at most τ_0 which is totally minimal with respect to the relative topology;

ii) there exists a dense subgroup of G with cardinality, less than $\max(\mathbf{c}, 2^\tau)$, which is totally minimal with respect to the relative topology;

iii) there exists a dense subgroup of G with cardinality at most τ_0 which is minimal with respect to the relative topology;

iv) there exists a dense subgroup of G with cardinality, less than $\max(\mathbf{c}, 2^\tau)$, which is minimal with respect to the relative topology;

v) the set of minimal non-zero closed subgroups of G has a cardinality at most τ_0 and for every prime p the group G does not contain copies of $\mathbf{Z}_p^{\tau+2}$;

vi) the set of minimal non-zero closed subgroups of G has a cardinality, less than $\max(\mathbf{c}, 2^\tau)$, and for every prime p the group G does not contain copies of $\mathbf{Z}_p^{\tau+2}$;

vii) there exists an exact sequence

$$0\leftarrow (T^1)^\alpha \xleftarrow{\lambda} G \xleftarrow{\mu} \prod_{p\in \mathrm{P}} (\mathbf{Z}_p^{\tau_p}\times F_p^*)\leftarrow 0,$$

where α and $\tau_p (p\in \mathrm{P})$ are cardinals, $\alpha<\max(\mathbf{a}_0, \tau)$, $\tau_p<\tau+2$, F_p denotes a p-group without non-zero divisible subgroups such that card $F_p<\max(\mathbf{a}_0, \tau)$, λ and μ are continuous group homomorphisms.

It should be noticed that Proposition 3 is analogical to Proposition 4 in [3].

Theorem 4. For every precompact minimal Abelian group G there exists a precompact totally minimal Abelian group G_1 and a topologically imbedding $i: G\to G_1$ such that $i(G)$ is dense in G_1 and card $G_1=$card G.

Proof. The statement follows directly from Proposition 3.

Let G be a compact group. By $m(G)$ we denote the minimal cardinality of dense subgroups of G, which are minimal with respect to the relative topology. Obviously, $m(G)\leq$card G.

Now we shall note some properties of $m(G)$.

Proposition 5. For every compact Abelian group G

$$\text{card } G^*\leq m(G),\ \text{card } G\leq 2^{m(G)}. \tag{1}$$

Proof. If G is finite (1) is trivial. We assume that G is infinite and put $\tau_0=m(G)$. If τ is the minimal cardinal with the property $\max(\mathbf{c}, 2^\tau)>\tau_0$, from the equivilence of iii) and vii) in Theorem 2, we find an exact sequence

$$0\to \mathbf{Z}^{(\alpha)}\to G^*\to \bigoplus_{p\in \mathrm{P}} (\mathbf{Z}(p^\infty)^{(\tau_p)}\oplus F_p)\to 0,$$

where $\alpha<\max(\mathbf{a}_0, \tau)$, $\tau_p<\tau+2$, card $F_p<\max(\mathbf{a}_0, \tau)$ $(p\in \mathrm{P})$.

It is easy to see now that card $G^* \leqq \tau_0$, and hence card $G \leqq 2^{\tau_0}$.

Proposition 6. For every infinite compact Abelian group G there are at least $m(G)$ closed subgroups in G.

Proof. The statement follows from the fact that for every infinite Abelian group X the set of all subgroups of X has a cardinality at least card X.

Example 1. Let $G = \prod_{p \in P} \mathbf{Z}_p$. Then $m(G) = \mathbf{a}_0$. It is easy to see that there are $\mathbf{c}$ closed subgroups in G. Therefore, the set of all closed subgroups of G has a cardinality, greater than $m(G)$.

Example 2. Let $G = T^n$. Then $G^* = \mathbf{Z}^n$, and hence the set of all closed subgroups of G is countable. On the other hand $m(G) = \mathbf{a}_0$. Therefore, the set of all closed subgroups of G has a cardinality $m(G)$.

Proposition 7. If H is a closed subgroup of the infinite compact Abelian group G, then

$$m(H) + m(G/H) \leqq m(G). \tag{2}$$

Proof. There exists a totally dense subgroup G' of G such that card $G' = m(G)$. Since $H \cap G'$ is totally dense in H, we find $m(H) \leqq m(G)$. If $j: G \to G/H$ is the canonical epimorphism, $j(G')$ is totally dense in G/H. Therefore, $m(G/H) \leqq m(G)$, which proves (2).

The following example shows that in (2) the equality may not hold.

Example 3. Let $G = \mathbf{Z}_p^2$, $H = \mathbf{Z}_p \times 0$. Now $m(H) = m(G/H) = \mathbf{a}_0$, but $m(G) = \mathbf{c}$. Hence $m(H) + m(G/H) < m(G)$.

Proposition 8. Let G be a compact Abelian group and H be a closed subgroup of G such that $\max(m(H), m(G/H)) \geqq \mathbf{c}$. Then $m(H) + m(G/H) = m(G)$.

Proof. Let $\tau_0 = \max(m(H), m(G/H))$ and τ be the minimal cardinal with the property $2^\tau > \tau_0$. We denote $X = G^*$, $\Gamma = H^\perp \subset X$. Then $\Gamma = (G/H)^*$ and $X/\Gamma = H^*$. Let A be a free subgroup of X such that X/A is periodic. We put $B = A \cap \Gamma$. Since B is free, we are able to find a free subgroup A_1 of A such that A/A_1 is periodic and $A_1 = B \oplus C$. It follows from Theorem 2 that there exist exact sequences

$$0 \to B \xrightarrow{i'} \Gamma \xrightarrow{j'} \bigoplus_{p \in P} T'_p \to 0,$$

$$0 \to A_1/B \xrightarrow{i''} X/\Gamma \xrightarrow{j''} \bigoplus_{p \in P} T''_p \to 0,$$

where i' and i'' are the canonical imbeddings, T'_p and T''_p are p-groups, card $T'_p < \tau$, card $T''_p < \tau$ $(p \in P)$.

Consider the following exact sequence

$$0 \to \Gamma/B \xrightarrow{i} X/B \xrightarrow{j} X/\Gamma \to 0,$$

where i and j are the canonical homomorphisms. Using that $(A_1 + \Gamma)/B = A_1/B \oplus \Gamma/B$ we find that the sequence

$$0 \to A_1/B \oplus \Gamma/B \xrightarrow{i''\oplus i} X/B \xrightarrow{j''\circ j} \bigoplus_{p\in P} T''_p \to 0$$

is also exact. Since $\Gamma/B = \bigoplus_{p\in P} T_p'$, there exists another exact sequence

$$0 \to \bigoplus_{p\in P} T'_p \to X/A_1 \to \bigoplus_{p\in P} T''_p \to 0.$$

Let $X/A_1 = \bigoplus_{p\in P} T'''_p$, $X/A = \bigoplus_{p\in P} T_p$, where T'''_p and T_p are p-groups $(p\in P)$ Then for every prime p card $T'''_p \leqq (\text{card } T'_p)(\text{card } T''_p) < \tau$ and hence card. $T_p < \tau$, because $X/A = (X/A_1)/(A/A_1)$. From the equivilence of iii) and vii) in Theorem 2 and from the fact that rank $A = \text{rank } \Gamma + \text{rank } (X/\Gamma)$ we find $m(G) \leqq \tau_0$. Therefore $m(H) + m(G/H) = m(G)$.

Corollary. If G is a compact Abelian group and card $G > \mathbf{c}$ then $m(H) + m(G/H) = m(G)$ for every closed subgroup H of G.

Proof. It is sufficient to prove that max $(m(H), m(G/H)) \geqq \mathbf{c}$. If we assume $m(H) < \mathbf{c}$ and $m(G/H) < \mathbf{c}$ it would follow from Proposition 3 that $m(H) \leqq \mathbf{a}_0$ and $m(G/H) \leqq \mathbf{a}_0$. Now from (1) we will find card $H \leqq \mathbf{c}$ and card $(G/H) \leqq \mathbf{c}$, which is impossible.

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ЕДНО СВОЙСТВО НА ПРЕДКОМПАКТНИТЕ МИНИМАЛНИ АБЕЛЕВИ ГРУПИ

Л. С т о я н о в

(РЕЗЮМЕ)

В тази работа се изучават някои кардинални инварианти на компактните абелеви групи и се доказва, че всяка предкомпактна минимална абелева група топологически се влага в предкомпактна тотално минимална абелева група със същата мощност. Методите, използвани тук, са развити в [3]. Главното твърдение в работата е теорема 2, която представлява обобщение и разширение на теорема 3 от [3].