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Interpolation Spaces and Some Geometric Constants

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Contents

Introduction

Interpolation theory and geometry of Banach spaces are well established and useful parts of functional analysis. These two theories support each other in various ways and many applications have been obtained as an outcome of their interplay.

The work consists of three sections. The first one deals with some notions from geometry of unit sphere $S(X)$ and unit ball $B(X)$ of a Banach space X. The second deals with real interpolation spaces, defined for $n-$tuples of Banach spaces and also for infinite families of Banach spaces and with some kind of operators acting in such families. The third section deals with Edmunds-Triebel logarithmic spaces, which are obtained from a couple first by complex interpolation and then by extrapolation. We consider measure of weak noncompactness of operators acting in such spaces and some geometric properties of this spaces.

For example, we consider the James constant $J(X)$, the Jordan-von Neumann constant $C_{NJ}(X)$, the Gao constants $E(X)$, $f(X)$, the constant $A_2(X)$, introduced by Baronti, Casini and Papini, the Zbăganu constant $C_Z(X)$, the n-th James constants $J_n(X)$ and the n-th Khintchine constants $K^n_{p,q}(X)$. The most significant are the James constant $J(X)$ and the Jordan-von Neumann constant $C_{NJ}(X)$. They are connected for instance to the so called fixed point property. The *James non-square constant* of a Banach space X is the number

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in B_X\},$$

and the *Jordan-von Neumann constant* $C_{NJ}(X)$ of X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \;:\; x, y \in X \text{ not both zero}\right\}.$$

These constants have been studied by several authors (Casini-1986, Gao-Lau 1990, 1991, Kato-Maligranda-Takahashi [30]-2001).

In the first subsection 1.1 called "Preliminaries" we give some background about $C_{NJ}(X)$ and $J(X)$ and relations between them, the connections to the constants $E(X), f(X), A_2(X), C_Z(X)$ and we give also some facts about notion of type and cotype of a Banach space, Khintchin constant, Clarkson inequalities.

In the subsection 1.2, called "B-convexity and n-th James constants" we deal with the notions of the uniformly non-l_1^n and B-convexity of a Banach space X. These notions are coming from James (1964) and Beck (1962).

For every natural number $n \geq 2$ we say, follow Giesy-James (1973), that a Banach space X is *uniformly non-l_n^1* if there exists an $\delta \in (0,1)$ such that for every $x_1, \ldots, x_n \in B_X$ it holds $\|\sum_{k=1}^n \theta_k x_k\| \leq n(1-\delta)$ for some choice of signs $\theta_1, \theta_2, \ldots, \theta_n$.

A Banach space X is called *B-convex* if it is uniformly non-l_n^1 for some $n \geq 2$.

In the connection to these two notions *the measure of uniformly non-*l_n^1 or sometimes called *the measure of B-convexity* appeared, i. e. for given $n \in \mathbf{N}$ *the n-th James constants* $J_n(X)$ of a Banach space X are the numbers defined by

$$J_n(X) = \sup\left\{ \min_{\theta_k=\pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\| \ : \ x_1, \ldots, x_n \in B_X \right\}.$$

Note that $J_1(X) = 1$, $J_2(X)$ is just the James constant $J(X)$ discussed in subsection 1.1 and $J_n(l^1) = J_n(l_m^1) = n$ for $m \geq n$.

It is clear that X is uniformly non-l_1^n if and only if $J_n(X) < n$, and X is B-convex if and only if $J_n(X) < n$ for some $n \geq 2$. Moreover, X is B-convex if and only if $\lim_{n\to\infty} \frac{J_n(X)}{n} = 0$.

The $n-$th James constants were studied by several authors e. g. Giesy in 1969, Pisier- 1974, Woyczynski- 1978, Kalton- 1978, Kalton- Peck- Roberts- 1984, Kadets-Kadets-1991, 1997 and Diestel- Jarchow- Tonge- 1995. It seems that these constant for the first time explicitely appeared in a paper by Giesy in 1966 (for the references see [43]).

Consider the *n-th strong James constants* of a Banach space X defined by

$$J_n^s(X) := \sup\left\{ \min_{\theta_k=\pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\| \ : \ x_1, \ldots, x_n \in S_X \right\}.$$

Then obviously $J_n^s(X) \leq J_n(X)$ and $J_3^s(X) < J_3(X)$ for $X = l_2^\infty$, that is, when $X = \mathbb{R}^2$ with the norm of $x = (x_1, x_2)$ equal to $\|x\| = \max\{|x_1|, |x_2|\}$. In fact, $J_3^s(l_2^\infty) = 1$ and $J_3(l_2^\infty) \geq 2$.

We don't know any example of a Banach space X such that dim$X \geq 3$ and $J_3^s(X) < J_3(X)$ but we guess that only $J_2^s(X) = J_2(X)$ for dim$X \geq 2$ and the conjecture is that

$$J_n^s(X) < J_n(X) \text{ for } n \geq 3 \text{ and } \dim X \geq 3.$$

Note, that $J_n^s(l^1) = J_n^s(l_m^1) = n$ for $m \geq n$ and for the Cesàro sequence spaces $ces_p, 1 < p \leq \infty$ in 2007 it was proved by L. Maligranda, N. Petrot and S. Suantai that $J_n^s(ces_p) = n$ for all natural $n \geq 2$, which means that they are not B-convex.

In subsection 1.2 we give some properties of these constants, for the proof see [43], here we give only some comments, connected to the use of notion of finitely representation. We prove reults about the $n-$th James constant for L_p spaces.

In subsection 1.3 we comment properties of the Khintchin (type) constant $K_{p,q}^n(X)$. Our main result here is an estimate of the $n-$th Khintchin constants by the $n-$th James constants. Is is formulated in Theorem 1.6 and it is written as it is in [43] , but it was given in slightly different form still in [55]. As a corollary from it we get some results about the connection between Jordan-von Neumann constant $C_{NJ}(X)$ and James constant $J(X)$, which was already proved in our work [54] in 2003. (This result itself improves the result from 2001 of Kato, Maligranda, Takahashi , [30].) Let us note, that the next step of improving the estimate of $C_{NJ}(x)$ by $J(X)$ was done by Takahashy and Kato in 2009.

We also continue a result of Giesy and James, (proved by them in 1973 for p=1) to the case $1 \leq p < 2$. It concerns the finitely representability of l_p in X.

Next we consieder the connection of $J_n(X)$ with the following characteristics $p(x) = \sup\{p \geq 1 : X \text{ is of type } p\}$, namely $\lim_{n\to\infty} \frac{\ln n}{\ln J_n(X)} = \sup_{n\geq 2} \frac{\ln n}{\ln J_n(X)} = p(X)$.

At the end of this subsection for the space $L_p[0,1]$ with a special norm the $n-$th James and Khintcin constants are calculated.

In the last subsection 1.4 we turn back to the case $n = 2$, i. e. to the classical James and Jordan - von Neumann constants. We consider also the Gao constants $E(X)$, $f(X)$, constant $A_2(X)$, introduced by Baronti, Casini and Papin and the Zbăganu constant $C_Z(X)$. From year 2000 some Japanese mathematicians (Kato, Saito, Mitani, Takahashi, . . .) began to use the idea of Bonsal and Dunkan from 1973, who showed that there is a correspondence between the set of absolute normalized norms N_a on C^n and a family Ψ_n of convex functions with special properties. For instance, if $n = 2$, Ψ_2 denotes the family of all convex functions ψ on $[0,1]$ with $\psi(0) = \psi(1) = 1$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $0 \leq t \leq 1$.

There are some examples of spaces with absolute normalized norms $\|.\|_\psi$, corresponding to different functions ψ, for instance l_p-norms $\|.\|_p$ correspond to the convex functions $\psi_p(t) = [(1-t)^p + t^p]^{1/p}$, $1 \leq p < \infty$. For $p = \infty$, the function $\psi_\infty(t) = \max\{t, 1-t\}$ corresponds to the norm $\|.\|_\infty$.

In the paper [44] Mitani and Saito gave a characterization of the James constant of an absolute norm on $\mathbf{C}^2$. As a consequence of their theorem, Mitani and Saito obtained result for function ψ which is comparable with ψ_2.

In subsection 1.4 subjects of investigation are two norms $\|.\|_\psi$ and $\|.\|_{\psi_*}$ with comparable corresponding functions ψ and ψ_*. We obtain theorems which give us relations between the considered constants in the case when $\psi \leq \psi_*$ or $\psi \geq \psi_*$ and generalize results of Mitani-Saito's from 2003 and Cui-Wang's [20] from 2007. Our results intersect with results from [71] from 2011 where James type constants are considered. In this subsection we give some of the results without proof (the proofs are in [60], from 2012), some theorems are with proofs and for some theorems we give ideas of how we act in [60] proving them. In this paper there are also results for means of two norms. We give the application of our results for some concrete norms, for instance for the norm $\max\{\|.\|_p, \lambda\|.\|_q\}$ which is shortly denoted by $\|.\|_{p,q,\lambda}$, where $1 \leq q \leq p \leq \infty$ and $\lambda \in (2^{\frac{1}{p}-\frac{1}{q}}, 1)$. For some particular cases of these norms, constants were calculated. For example, in [30] constants J and C_{NJ} are calculated for the norm $\|.\|_{\infty,p,\frac{1}{\lambda}}$; the constant C_{NJ} is obtained for norms $\|.\|_{2,1,\lambda}$ and $\|.\|_{2,\infty,\lambda}$ in [64]. Some general properties of the norm $\|.\|_{p,q,\lambda}$ are discussed in our paper [59]. Here we consider constants J, C_{NJ}, C_Z, E, f and A_2 for the norm $\|.\|_{p,q,\lambda}$. In different cases depending of how λ, p and q are situated we have different success. In some we can calculate the constants, in other we only have some estimates, for instance an open problem is calculation of the constant in the case $1 \leq q < 2 < p$, $\lambda \in (2^{\frac{1}{p}-\frac{1}{2}}, 2^{\frac{1}{2}-\frac{1}{q}})$. We consider the same things for the norm $\|.\|_{\alpha,p,q,r} = (1+\alpha)^{-\frac{1}{r}}(\|.\|_q^r + \alpha\|.\|_p^r)^{\frac{1}{r}}$. In particular, if $p = \infty, q = 2, r = 2$, our results covers results of Example 7 from [30] for the $J(\|.\|_{\psi_{\alpha,p,q,r}})$ and $C_{NJ}(\|.\|_{\psi_{\alpha,p,q,r}})$ and results from [20] for $E(\|.\|_{\psi_{\alpha,p,q,r}})$ and $f(\|.\|_{\psi_{\alpha,p,q,r}})$. See also Example 2. 8 from [71] in the case of $A_2(X)$ and $E(X)$. When p, q are numbers from $[1, \infty)$, but $r = 2$, then the James and von Neumann-Jordan constants are calculated in a paper by C. S. Yang and H. Li in 2010, using other methods. In the very recent paper [45] -2011, the case $r = 2$ was consiedered and results for C_{NJ}, C_Z and $C'_{NJ} = \frac{1}{4}E$ are obtained.

At the end of the subsection we consieder two groups of norms: norms of X^p spaces and the Cesàro norm and related norms.

We have to mention that idea of working with convex functions instead of norms helps also in other situations, for instance , finding the dual space, proving some inequalities. We do not include here results from our papers about some inequalities (obtained using such approach), written recently in cooperation with S. Varoshanec [59] and with S. Varoshanec and L. -E. Persson [53]. But this can be also an excuse for writing this subsection, which deals with the above correspondance between norms and convex functions. Let us mention, that we consider also some type of concave functions in the just mentioned papers.

In the theory of interpolation (see [6]) one usually considers Banach couples, i.e. pairs (A_0, A_1) such that A_0 and A_1 are Banach spaces embedded in a common topological vector space U. The most important among the various constructions of interpolation with respect to a given couple is the complex method leading to the spaces $[A_0, A_1]_\theta$ (where $0 < \theta < 1$) and the real method leading to the spaces $(A_0, A_1)_{\theta,q}$ (where $0 < \theta < 1$ and $0 < q \leq \infty$). Usually the Banach couple is denoted by $\overline{A} = (A_0, A_1)$, the intersection $\Delta\overline{A} = A_0 \cap A_1$ is provided with the norm $\|a\|_{\Delta\overline{A}} = \max(\|a\|_{A_0}, \|a\|_{A_1})$, the sum $\Sigma\overline{A} = A_0 + A_1$ consists of those elements of U, which can be represented as $a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1$ and its norm is $\|a\|_{A_0+A_1} = \|a\|_{\Sigma\overline{A}} = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + \|a_1\|_{A_1})$.

We say that A and B are interpolation spaces of type θ if

$$\|T\|_{A\to B} \leq C(\|T\|_{A_0\to B_0})^{1-\theta}(\|T\|_{A_1\to B_1})^{\theta},$$

if C=1, i.e. we have log-convexity inequality, then we say that A and B are exact interpolation spaces of type θ. At the beginning of second section we give definition of complex and real methods of interpolation for couples of Banach spaces, here we mention mainly the real K-method.

The Peetre K-functional is defined like

$$K(t,a) = K(t,a,\overline{A}) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}),\ a \in \Sigma\bar{A} = A_0 + A_1.$$

The interpolation space, constructed by K-method is

$$A_{\theta,q,K} = \{a \in \Sigma\overline{A} : \|a\|_{\theta,q,K} = \Phi_{\theta,q}(K(t,a)) = \left(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{dt}{t}\right)^{1/q} < \infty\}$$

for admisssible values of θ and q.

When define the space of real J-method one works not with $\Sigma\overline{A}$, but with $\Delta\overline{A} = A_0 \cap A_1$ and the J-functional.

Let us note that the complex method, K-method and J-method give exact interpolation spaces of type θ.

Parts of the theory concerning interpolation between two Banach spaces can be generalized to cover also the cases where one interpolates between finitely many Banach spaces and even between general families of (infinite many) Banach spaces. We begin subsection 2.1.2 with some history about this area: some information about the development of complex interpolation and much more details about real methods. For the case of finite family we mention Sparr's method, Fernandez's method and Cobos-Peetre methods, and for the infinite family the "continuous" method, suggested with M. J. Carro and "discrete method" suggested by L. -E. Persson and me. Note that by a suggestuon of J. Peetre to join our efforts in 1997 our paper [14] , (by J. Peetre, M. J. Carro, L. -E. Persson and me) appeared. The collaboration continued and some paper, with coauthors M. J. Carro and me, and in some also L. -E. Persson appeared - see [12], [13] [15].

In our background we mention some application of complex method for infinite families, for instance, Ferenczi [26] provides an example of a complex uniformly convex hereditarily indecomposable space. In [63] Pisier used also the idea of complex interpolation in families of Banach spaces to get some results from geometry of Banach spaces.

Later we mention also some cases in which real interpolation of finite families can have nice applications. Begin with the three-space approach, the interpolation in triples of Banach spaces can be interesting for instance in the case of smooth functions, see [3]. Moreover, results about interpolation of triples of weghted Lebesque spaces and a variant of reiteration theorem is used to extend the Stein-Weiss interpolation theorem, known for $L_p(\mu)$-spaces with change of measures to Lorentz spaces wit change of measure. In particular this shows that for some problems in analysis the three-space real approach is really more useful than the usual real interpolation between couples.

As an important consequence of our results about interpolation in infinite families of Banach spaces a new proof can be given of so called one-side compactness result for couples of Banach spaces, which has remained an open question for several years till M. Cwikel gave in 1992 a positive answer. With our tecniques we can give the insight of this particular problem.

When speaking about interpolation of (infinite) families of Banach spaces we have to mention the complex methods (see [19]), [34] and [35]), they appear before the real methods. In terms of [19], let denote complex interpolation space $A(z)$, $|z| < 1$, constructed for the family $\bar{A} = \{A(\gamma),\ \gamma \in \Gamma\}$, and the log-intersection of the family as $\mathtt{A} = \{a \in A(\gamma) \text{ for a.e. } \gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma \, dg < \infty\}$.

The interpolation theorems say that if the linear map T acts from $\Sigma\bar{A}$ to $\Sigma\bar{B}$ and its restrictions $T/A(\gamma)$ is bounded operator from $A(\gamma)$ to $B(\gamma)$, T maps $\mathtt{A}$ to $\bigcap B_\gamma$ and $\|Ta\|_{B_\gamma} \le M(\gamma)\|a\|_{A_\gamma}$ then T maps the interpolation space $A(z)$ into $B(z)$ and the norm there is less or equal $M(z) = \exp\big(\int_\Gamma \log \ M(\gamma)\, P_z(\gamma) d\gamma\big)$, P_z being the Poisson kernel. This can be regarded as an infinite variant of the inequality (log convexity inequality) in the notion of exact interpolation method of type θ (here z) in the case when the families consist of just two spaces (the case of Banach couples).

The situation in the real interpolation for infinite families is more complicated and the norm of the operator, acting between the interpolation spaces is estimated by the so called Dicesar function. This function in some situations can be calculated and in some special cases is equal to $M(z)$. Nicer is the situation when the family is an $n-$ tuple, i.e. it consists of n Banach spaces.

In 1988, in [49] for $q = 1$ I considered a discrete K-functional

$K_q^{(1)}(\alpha, a) = \inf \left\{ \left(\sum_j (\alpha(\gamma_j) \|a_{\gamma_j}\|_{A_{\gamma_j}})^q \right)^{1/q} \right\}$, where the infimum is taken over all representations of the element $a = \sum_j a_{\gamma_j}$ with convergence in U and $a_{\gamma_j} \in A(\gamma_j)$.

Another K-functional was defined by M. J. Carro in 1994 (see [10], [11]) as

$K_q^{(2)}(\alpha, a) = \inf\left\{\left(\int_\Gamma (\alpha(\gamma)\|a(\gamma)\|_\gamma)^q\, d\gamma\right)^{1/q}\right\}$, and the infimum extends over all representations $a = \int_\Gamma a(\gamma)\, d\gamma$ (convergence in U) with $a(\cdot) \in \bar{G}$ where $G = \left\{b = \sum_{\text{finite}} b_j \chi E_j : b_j \in \mathtt{A} \text{ and } E_j \text{ pairwise disjoint measurable sets in } \Gamma\right\}$.

Let S be a set of measurable functions $\alpha : \Gamma \to R^+$. We will suppose that it is a multiplicative group of bounded (in the discrete case) and essentialy bounded (in the continuous case) functions.

For $1 \le p, q \le \infty$, the following interpolation spaces were defined in [14]: $(A)^{S,j}_{z_0,p,q;K} = \left\{a \in U;\ \left(\sum_{\alpha\in S}\left(\frac{K_q^{(j)}(\alpha, a)}{\alpha(z_0)}\right)^p\right)^{1/p} < +\infty\right\}$. In case $j = 1$ this is the space of "discrete" case, if $j = 2$ this is the "continuous" case.

When instead of infinite variants of K-functional we use infinite variants of J-functional we get $(A)^{S,j}_{z_0,p,q;J}$. We give only some facts from [14], for instance we show, that for any bounded interpolation family $\overline{A}$ one has $(A)^{S,2}_{z_0,p,q;K} \subset (A)^{S,1}_{z_0,p,q;K}$. We show also, that if we restrict ourselves to countable families, then $(A)^{S,1}_K = (A)^{S,2}_K$. The subsection finishes with considering particular cases of S when we have finite families: Sparr's, Fernandez's spaces, Cobos method of polygons and its generalization for the infinite families, suggested by me first in 1991 [48] and developed in my paper [52] with L. -E. Persson in 1996.

The next subsection is devoted to the Dicesar function $D^S_{z_0}(M) = \inf_{\alpha\in S}\left\{\frac{\sup \alpha(\gamma)M(\gamma)}{\alpha(z_0)}\right\}$, where S is a subset of $\widetilde{L} = \{\alpha : \Gamma \to R^+, \log\alpha \in L^1(\Gamma)$, $z_0 \in D$ and $M \in \widetilde{L}$.

If F is either $(\cdot)^S_{z_0;K}$ or $(\cdot)^S_{z_0;J}$ and if S is a multiplicative group, then for $\bar{A}$ and $\bar{B}$ - two i.f. (interpolation families) and $T : \overline{A} \to \overline{B}$ - an interpolation operator such that $\|T\|_{A(\gamma)\to B(\gamma)} \le M(\gamma)$ a. e. $\gamma \in \Gamma$ with $M \in \widetilde{L} = \{\alpha : \Gamma \to R^+, \log\alpha \in L^1(\Gamma)\}$, then (see [11], [14]) we have that $\|T\|_{F(\bar{A})\to F(\bar{B})} \le D^S_{z_0}(M)$.

This function was introduced in [14] as the Dicesar function, since the authors found it for the first time in the paper of Dicesar Fernández [26]. This function is a generalization of the function $M_0^{1-\theta}M_1^\theta$ for the classical case of two Banach spaces. Other special cases of the Dicesar function are $D_{(\alpha,\beta)}(M_1, \cdots, M_n)$, suggested by Cobos and Peetre in [18] in the finite family case and its generalization $D_\theta(M)$ for infinite family, suggested by me and L. E. Persson in [52]. We consider also a function, called $E_{(\alpha,\beta)}(M_1, \cdots, M_n)$, in Theorems 1. and 2. in my paper [48] where the measures of non-compactness of operators acting in $n-$tuples of Banach spaces are estimated. Since $1 \in S$, $D^S_{z_0}(M) \le \|M\|_\infty$ and obviously $D^S_{z_0}(M) \ge M(z_0)$. Moreover, if $M \in S$, $D^S_{z_0}(M) = M(z_0)$. Finally, the connection between the function $D_{z_0}^S(M)$ and the classes $K_{z_0}^S$ and $J_{z_0}^S$ was studied in [14], where also some concrete examples of the Dicesar function were computed.

The Dicesar function plays a fundamental role in our papers [13] and [15]. Here we will consider the case when S is not a group and also a new function $D^S_{z_0,p;J\to K}(M)$ that give us an upper estimate for the norm of the interpolation operator when acting from $(\bar{A})^S_{z_0,p;J}$ into $(\bar{B})^S_{z_0,p;K}$.

The Theorem proved in this subsection is used in our paper with M. J. Carro [13] about interpolation of compactness property, namely to prove so called one sided compactness result, where the condition about compactness of the operator is only on a subset γ, but not on the whole Γ. We (for shortness) are going to write here only the estimate for the case $p < 1$, namely

$$\|T\|_{(\bar{A})^{S_1}_{z_0,p;J}\to(\bar{A})^{S_2}_{z_0,p;K}} \leq \left(\sup_{\beta\in S_1}\sum_{\alpha\in S_2}\left(\frac{\inf_{\gamma\in\Gamma}(\alpha\beta^{-1}M)(\gamma)}{\alpha(z_0)\beta^{-1}(z_0)}\right)^p\right)^{1/p}.$$

More about the Dicesar function and related results can be found in our paper with M. J. Carro and L.-E. Persson [15] where the Dicesar function controls the norms of some interpolation operators, in particular some non-linear operators, operators of weakened type (A,Ψ), correct operators acting in some Banach lattices, summing operators. Yet in [51] we have a result about Dicesar function for C-subadditive operators in Cobos-Peetre spaces. The Dicesar function is also involved when we estimate the measure of noncompactness - [48]. In subsection 2.2 we prove some Lions-Peetre type interpolation results (when one of the family $A(\gamma)$ or $B(\gamma)$ is constant) - about compact, limited and of weakened type operators and also about measure of noncompactness [50]. Here the classes $K(A,Z)$ and $J(A,Z)$ introduced by Cwikel and Janson in [21] appear. We do not include here results about weakly compact operators from paper [12] because of the big volume of this paper and because we will consider measure of weak non-compactness results in the third section. We get results of following type:

Let $A_t, t\in\Gamma$ be a bounded family of Banach spaces, B be an arbitrary Banach space and let $A\subset K(A,Z)$. Let γ be a subset of Γ with positive measure. Suppose that $T:\sum A_t\to B$, $\sup\|T/A_t\|_{A_t\to B_t}<\infty$ and T is a limited operator from $\sum_\gamma A_t$ into B. Then T is a limited operator, acting from A into B.

Similar results we get for the compact and weakened type operators, see Theorem 2.6, corollary 2.9 and Theorem 2.12. Note that Theorem 2.8 gives an estimate about measure of non-compactness for the interpolatin operator of Lions-Peetre type. Remember, if $k\geq 0$, then a map $T:A\to B$ is called a k-set of contraction iff $\widetilde{\psi}_B(T(E))\leq k\widetilde{\psi}_A(E)$ for all bounded sets E and $\tilde{\beta}(T)=\min\{k: T$ is a k-set contraction$\}$ is called the measure of noncompactness of T, where $\widetilde{\psi}_A(E)$ is the Hausdorff measure of non-compactness of the set E in the space B. Another measure of noncompactness of an operator $\beta(T)$ is defined analoguously to $\tilde{\beta}(T)$ using the Kuratowski measure of non-compactness ψ_A and ψ_B.

To extend this result to the case when both families are not constant an often used approach (due to Arne Persson) is to put some approximation conditions. We also put such a hypothesis on the $n-$tuple $\bar{B}$ and get that if A and B are interpolation spaces defined by polygon method and if $T \in L(\bar{A}, \bar{B})$, the following inequality holds

$$\tilde{\beta}(T_{A\to B}) \leq D_{(\alpha,\beta)}\left(C_1\tilde{\beta}(T_{A_1\to B_1}), C_2\tilde{\beta}(T_{A_2\to B_2}), \dots, C_n\tilde{\beta}(T_{A_n\to B_n})\right)$$

(here $C_1, \dots, C_n$ are the constnts which appear in the hypothesis).

Since $\tilde{\beta}(T) = 0$ (which is the same $\beta(T) = 0$) means that T is compact operator, from the above we get a result of interpolation of compactness property.

In the last subsection we deal with some results from the paper [3], written by 5 authors. It is a long paper - 36 pages. My personal contribution to this work is mainly in the proof of the second reiteration formula and in the interpolation of block- Lorentz spaces. In [3] these results and a wavelet approach for (θ, q) spaces for triples of smooth function spaces (such as Besov spaces, Sobolev spaces) are used. In contrast to the case of couples, for which even the scale of Besov spaces is not stable under interpolation, for triples we obtain stability in the frame of Besov spaces based on Lorentz spaces. Another application of the analogue of the Lions-Peetre reiteration theorem for triples of Banach function lattices is connected with the Stein-Weiss interpolation theorem known for L_p spaces with change of measures. This theorem is extended to Lorentz spaces with a change of measures. In particular, results obtained in [3] show that for some problems in analysis the three-spaces real interpolation approach is realy useful. Now about what is the subsection about triples of Banach function lattices.

In 1964 Lions and Peetre proved one of the most important theoretical results in interpolation theory, the so called reiteration formula for couples of Banach spaces:

$$(\bar{X}_{\theta_0,q_0}, \bar{X}_{\theta_1,q_1})_{\lambda,q} = \bar{X}_{\theta,q}, \quad \theta = (1-\lambda)\theta_0 + \lambda\theta_1$$

where $\theta_0 \neq \lambda\theta_1$.

This formula also holds for quasi-Banach spaces ([6], Th 3.11.5.) It shows not only the stability of the spaces $\bar{X}_{\theta,q}$, but it also gives a possibility to calculate the interpolation spaces for rather complicated couples $(\bar{X}_{\theta_0,q_0}, \bar{X}_{\theta_1,q_1})$ by using simpler initial couple $\bar{X} = (X_0, X_1)$ and the result does not depend on the parameters q_0 and q_1.

The classical proof of the reiteration formula is based on the equivalence theorem for the K- and J- methods, which is valid for any couple of quasi-Banach spaces, but not even for triples. Acting in another way, namely with the so called K-divisibility, is not admissible for triples, because the K-divisibility is not valid for triples (an example, by I. Asekritova). So we cannot expect success on such approaches for the very general situation. In 1997 Asekritova and Krugljak showed that equivalence theorem is in fact valid for any $n-$tuple of Banach function lattices on Ω so we decided in [3] to consider this case. To prove the second reiteration formula (for triples of quasi-Banach function lattices), namely

$$((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_2)_{\alpha_1,q_1})_{\mu,q} = (X_0, X_1, X_2)_{(\theta_1,\theta_2),q},$$

where $0 < q_0, q_1, q < \infty$ and $\frac{1}{q} = \frac{1-\mu}{q_0} + \frac{\mu}{q_1}$, $\theta_1 = (1-\alpha_1)\mu$, $\theta_2 = \alpha_0(1-\mu)+\alpha_1\mu$ we need first to prove lemma of independet interest about embeddings of the K and J methods, when all q's are equal to 1, and then to use the equivalence theorem of Asekritova-Krugljak and a power theorem.

Next we prove some interpolation theorems for so called block-Lorentz spaces, defined in the following way: Let ω be a weight function on Ω, the sets $\Omega_k = \Omega_k(\omega) = \left\{x \in \Omega : 2^k \leq \omega(x) \leq 2^{k+1}\right\}$, $k \in \mathbb{Z}$, and define, for $\sigma \in \mathbb{R}$ and $0 \leq p, q, r, \leq \infty$, the *block - Loretntz spaces* $L^{\sigma,q}_{p,r} = L^{\sigma,q}_{p,r}(\omega)$ by the finiteness of the quasi-norm $\|f\|_{L^{\sigma,q}_{p,r}} = \left(\sum_{k\in\mathbb{Z}}(\|f\omega^\sigma\chi_{\Omega_k}\|_{L_{p,r}})^q\right)^{1/q}$ with the standard modification for $q = \infty$. Here, $L_{p,r}$ denotes the usual Lorentz space (with the convention that $r = \infty$ when $p = \infty$). It is clear that $L^{\sigma,p}_{p,p} = L^\sigma_p$ and in what follows we also use the notation $L^{\sigma,q}_p := L^{\sigma,q}_{p,p}$.

In the case when $r = p$ and $\omega(x) = |x|$ on $\mathbb{R}^n \setminus \{0\}$, the spaces $L^{\sigma,q}_{p,r}$ are the so called homogenous Herz spaces $K^{\sigma,q}_p$.

The third section, (devoted to Edmunds- Triebel logarithmic spaces), in some sense gathers some topics from interpolation theory, extrapolation theory and some geometric and topological properties of the obtained subject, namely of logarithmic spaces.

Interpolation theory has serious applications to operator theory (see at least the monographs by Pietch [62] and Triebel [69]), but the connection of extrapolation theory with operator theory is not fully studied. Triebel used in 1993 extrapolation ideas for studying the degree of compactness of some limiting Sobolev embeddings, abstract results (for instance for special operator ideals, etc.) have Cobos, Kuhn, M. J. Carro, Fernandez-Cabrera, Martinez, A. Kryczka. Consider a construction, suggested by Edmunds and Triebel in [25] who defined for every $0 < \theta < 1$, $1 < q < \infty$, and $b \in \mathbb{R} \setminus \{0\}$ the logarithmic space $A_\theta(logA)_{b,q}$. These spaces can be regarded as a special case of extrapolation spaces. Roughly speaking (for the case $b > 0$), for a couple of Banach spaces (A_0, A_1) we construct the complex interpolation spaces $[A_0, A_1]_\theta$ (where $0 < \theta < 1$) and then we extrapolate with the so called Σ_q extrapolation method over $A(i) = 2^{ib}[A_0, A_1]_{\eta(i)}$ for $i \geq J$, where $J \in \mathbb{N}$ such that $\theta - 2^{-J} > 0$ and $\eta(i) = \theta - 2^{-i}$ for $i \geq J$. The construction for the case $b < 0$ uses complex interpolation spaces with other indexes, named $\theta(i)$ and Δ_q extrapolation method. The spaces $A_\theta(logA)_{b,q}$ form a scale, thinner than the scale of the complex method $[A_0, A_1]_\theta$. Note that the usual Zygmund space $L_p(LogL)_b(\Omega)$ (i. e. the set of all measurable functions $f : \Omega \to C$ such that $\int_\Omega |f(x)|^p \log^{bp}(2 + |f(x)|)dx < \infty$) is isomorphic to the logarithmic space $A_\theta(logA)_{b,p}$, where $A_0 = L_\infty(\Omega)$, $A_1 = L_1(\Omega)$ and $\theta = p^{-1}$ and we have an infinite family of equivalent norms (depending on J), see [25]). When X is a normed space with $\dim X \geq 2$ and $B_X = \{x \in X : \|x\| \leq 1\}$ is the unit ball of X, the modulus of convexity $\delta_X(\varepsilon)$ of X, for $0 \leq \varepsilon \leq 2$, is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \varepsilon\}.$$

The space X is said to be uniformly convex (u.c.) if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

First we estimate the modulus of convexity of the $p-$ sum and as a consequence we get that whenever $1 < p, k < +\infty$ if the spaces (X_n) have modulus of convexity of power type k uniformly, that is there is $0 < c < +\infty$ such that $\delta(X_n) \geq c\varepsilon^k$ for every $n \in \mathbf{N}$, then the space $[\sum_{n=1}^{\infty} X_n]_p$ has modulus of convexity of power type $\max\{p, k\}$.

As a consequence for the classical case we get that for $b < 0$ and $1 < p \leq 2$ (resp. $2 < p$), the Zygmund spaces have modulus of convexity of power type 2 (resp. p) for all of the eqiuvalent norms from [25]. For $b > 0$, if $p < 2$ then the Zygmund spaces have modulus of convexity of power type 2 for each of these norms; if $2 \leq p$ then they have modulus of convexity of power type r for every $r > p$, for an infinite number of these norms.

Given an interpolation couple (A_0, A_1), if at least one of A_0, A_1 is u. c. then the space A_θ is u. c. for every $0 < \theta < 1$ [22]. The converse does not hold. However we show a type of extrapolation result, namely if one of the interpolation spaces, let say A_θ is u. c. then all of them are. Moreover, we show that if there exists θ_0, $0 < \theta_0 < 1$, such that the complex interpolation space A_{θ_0} is u. c., then the logarithmic space $A_\theta(\log A)_{b,p}$ is u. c. for every $0 < \theta < 1$, $1 < p < \infty$ and $b \in \mathbf{R}\backslash\{0\}$.

When one of A_0 and A_1 or both of them are uniformly convex, we give an estimate for the moduli of convexity of the equivalent norms of $A_\theta(\log A)_{b,p}$ in terms of the moduli of A_0 and A_1. We give also such an estimate in terms of modulus of the space A_θ.

The next subsection is about the measure of weak noncompactness of Edmunds-Triebel spaces and B-convexity of Edmunds-Triebel spaces. We use here not the probably more popular De Blasi (1992) measure, but the one, defined by Kryczka, Prus and Szczepanik in 2000- $\gamma(M)$, M being bounded. We note here that $\gamma(M)$ coincides with the function measuring the deviation from relative weak compactness based on the double-limit criterion, considered by K. Astala and H. O. Tylli in 1990. Namely,

$$\gamma(M) = \sup\{\lim_{m\to\infty}\lim_{n\to\infty} f_m(x_n) - \lim_{n\to\infty}\lim_{m\to\infty} f_m(x_n) : \\ (x_n)_{n\in\mathbb{N}} \subset M,\ (f_m)_{m\in\mathbb{N}} \subset B_{X^*} \textit{ and the limits exist}\}.$$

Note also, that $\gamma(M) = 0$ iff M is relatively weakly compact.

For every bounded operator $T : E \to F$ the number $\Gamma(T) = \gamma(T(B_E))$ is called measure of weak noncompactness of the operator T. In [38] the number $\Gamma_\theta(T)$ of the operator acting between complex interpolation spaces with index θ is estimated, namely a log convexity inequality is obtained. Moreover, in [37] there are results for the extrapolation spaces. Combining this two results we get estimates for $\Gamma(T)$ acting in logarithmic spaces. As a corollary we get that if one of the operators $T : A_j \to B_j,\ j = 0,1$, is weakly compact, then the operator $T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}$ is also weakly compact. Another consequence:If A_0 or A_1 is reflexive, then the space $A_\theta(logA)_{b,q}$ is also reflexive.

Later on we decided to spread this idea for estimation of C^n_{NJ} and J^n of logarithmic spaces. For this purpose we first get estimates of C^n_{NJ} and J^n for the complex interpolation spaces, which can be of independent interest. For to get estimate of n-th James constant we use also a corrollary from the main theorem from our paper [43] and the estimate, which we prove for extrapolation spaces for $n-$th Jordan - von Neumann constant. For the case of classical $(n = 2)$ James constant we get simplier and sharper estimate. Let us mention, that as a corollary we get, that if one of the spaces A_0 and A_1 is uniformly non- ℓ^1_n, then the logarithmic space $A_\theta(logA)_{b,q}$ is uniformly non- ℓ^1_n (and of course) if one of the spaces A_0 and A_1 is B- convex, then the logarithmic space $A_\theta(logA)_{b,q}$ is B-convex.

Let $1 < p \leq 2$. A Banach space X satisfies the (p, p') Clarkson inequality (we write $CI(p, p')$ holds in X) if

$$(\| x + y \|^{p'} + \| x - y \|^{p'})^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} (\| x \|^p + \| y \|^p)^{\frac{1}{p}}$$

for every $x, y \in X$. It is obvious that the Clarkson inequality for $p = 1$ holds in every Banach space. If $CI(p, p')$ holds in X for some p, $1 \leq p \leq 2$, then $CI(r, r')$ holds in X for every r with $1 < r \leq p$.

First we prove some fact about Clarkson inequality in complex interpolation space and then about it in the logarithmic spaces. Here again for shortness we will mention only the classical example of Zygmund spaces, namely we mention that

i) If $b < 0$ and $1 \leq r < p$, then there exists an equivalent norm in $L_p(logL)_b(\Omega)$ such that this space satisfies $CI(r, r')$.
ii) If $b > 0$, then there exists an equivalent norm in $L_p(logL)_b(\Omega)$ such that this space satisfies $CI(p, p')$.

Combining the theorems about $CI(p, p')$ in $A_\theta(logA)_{b,q}$ with the facts about relations between type, cotype and Clarkson inequalities we get results about type and cotype of such spaces. The corollary for Zygmund spaces read

i) If $b > 0$, then a renorming of the spaces $L_p(logL)_b(\Omega)$ can be made such that in this norm (in these norms) these spaces and their duals are of Rademacher type p with type constant 1 and of Rademacher cotype p' with cotype constant 1.
ii) If $b < 0$, then a renorming of the spaces $L_p(logL)_b(\Omega)$ can be made in such a way that in this norm (in these norms) these spaces and their duals are of Rademacher type r with type constant 1 and of Rademacher cotype r' with cotype constant 1, for every $1 \leq r < p$. Some results by Kaminska and Turett show that this is in a sense the best possible.

In the first section we used results from the papers [43], [54], [55], [60]. We found some citations: 7 for [43] and 4 for [55]. The paper [60] has to appear in 2012. Papers [43] and [60] have Impact factor. In the second section we used rezults from the papers [3], [14], [15], [48] [50], [51] . Citations: 18 for [3], 5 for [14], 7 for [48],1 for [51]. Papers [3], [14], [48], [51] have Impact factor.

In the third section we used results from the papers [56], [57], [58]. Citations: 3 for [56], 1 for [57], 1 for [58]. All three papers have Impact factor.

1. Constants in Banach spaces

Several constants of a Banach space $X = (X, \|.\|)$ are used in the description of its geometric properties. For example, the James constant $J(X)$, the Jordan-von Neumann constant $C_{NJ}(X)$, the Gao constants $E(X)$, $f(X)$, the constant $A_2(X)$, introduced by Baronti, Casini and Papini, the Zbăganu constant $C_Z(X)$, the $n-$th James constants $J_n(X)$ and the $n-$th Khintchine constants $K^n_{p,q}(X)$. We will collect properties of these constants and relations between them.

In the first three subsections we will deal with the connections between the Jordan-von Neumann constant and the James constant and also between the $n-$th James constants and the $n-$th Khintchine constants. In the last subsection we deal with the constants $J(X),\ C_{NJ}(X),\ E(X),\ f(X),\ A_2(X),\ C_Z(X)$ in $\mathbf{C}^2$ equipped with different absolute normalized norms.

1.1. **Preliminaries.** A Banach space $X = (X, \|.\|)$ will be a real Banach space with $\dim X \geq 2$. B_X will denote the closed unit ball $\{x \in X : \|x\| \leq 1\}$ of our Banach space and $S_X = \{x \in X : \|x\| = 1\}$ is its unit sphere. The *James* non-square *constant* of a Banach space X is the number $J(X)$ defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in B_X\},$$

and the *Jordan-von Neumann constant* $C_{NJ}(X)$ of X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} :\ x, y \in X \text{ not both zero}\right\}.$$

These constants have been studied by several authors (Casini-1986, Gao-Lau 1990, 1991, Kato-Maligranda-Takahashi [30]). Let us collect some properties of these constants:

(1) $J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$.

(2) $\sqrt{2} \leq J(X) \leq 2$ *and* $1 \leq C_{NJ}(X) \leq 2$.

(3) X *is Hilbert space* $\Longrightarrow J(X) = \sqrt{2}$ *and the converse is not true;* $C_{NJ}(X) = 1 \Longleftrightarrow X$ *is a Hilbert space.*

(4) $J(X) < 2 \Longleftrightarrow C_{NJ}(X) < 2 \Longleftrightarrow$ *the space* X *is uniformly non-square, i.e., there exists a* $\delta \in (0,1)$ *such that for any* $x, y \in S_X$ *either* $\|x+y\|/2 \leq 1-\delta$ *or* $\|x-y\|/2 \leq 1-\delta$.

(5) $J(X^{**}) = J(X),\ \max\{\sqrt{2}, 2J(X)-2\} \leq J(X^*) \leq J(X)/2+1$, *and there exists a two-dimensional Banach space* X *such that* $J(X^*) \neq J(X)$, *where* X^* *and* X^{**} *are the dual and the second dual of* X; $C_{NJ}(X^*) = C_{NJ}(X)$.

(6) If $1 \leq p \leq \infty$ *and dim* $L^p(\mu) \geq 2$, *then* $J(L^p(\mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$ *and* $C_{NJ}(L^p(\mu)) = \max\{2^{2/p-1}, 2^{1-2/p}\} = 2^{|1-2/p|}$.

Kato-Maligranda-Takahashi, [30] proved that

$$\frac{J(X)^2}{2} \leq C_{NJ}(X) \leq \frac{J(X)^2}{(J(X)-1)^2+1}.$$

Moreover, if X is not uniformly non-square, then we have equalities in (1) and there exists a two-dimensional Banach space X for which $J(X)^2/2 < C_{NJ}(X)$.

We improved the right hand side inequality in 2003, [54], see also [55], [43], namely

$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}\left(\sqrt{J(X)^2 - 4J(X) + 8} - 2\right)$$

Maligranda even formulated the following conjecture :

$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1$. It stayed open for some years.

We will not present the proof of our inequality, because this estimate was improwed in 2009 by Kato and Takahashi and looks like $C_{NJ}(X) \leq J(X)$.

Very recently a paper [7], dealing with James constant for some real interpolation spaces for finite families of Banach spaces appeared.

In 2006 the Gao constants are defined like

$$E(X) = \sup\{\|x+y\|^2+\|x-y\|^2 : x, y \in S_X\}, f(X) = \inf\{\|x+y\|^2+\|x-y\|^2 : x, y \in S_X\}.$$

Sometimes instead of $E(X)$ the constant $C'_{NJ}(X) = \frac{1}{4}E(X)$ is considered. Another constants are

$$A_2(X) = \sup\{\frac{\|x+y\| + \|x-y\|}{2} : x, y \in S_X\},$$

$$C_Z(X) = \sup\{\frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} : |x\|^2 + \|y\|^2 \neq 0\}.$$

The onstant $A_2(X)$ was introduced in 2000 by M. Baronti, E. Casini, P.L. Papini, $C_Z(X)$ was introduced by Zbăganu, who conjectured that $C_Z(X)$ and $C_{NJ}(X)$ always coincide, which was disapproved by Alonso and Martin in 2006.

It is worthwhile to clarify the relation between the constants. Many authors have improved the known estimations, F. Wang, C. Yang, S. Saejung, H.Li, let us mention some of these results: $C_{NJ}(X) \leq J(X)$, $\frac{J^2(X)}{2} \leq C'_{NJ}(X) \leq C_{NJ}(X) \leq A_2(X) \leq \sqrt{2C'_{NJ}(X)} \leq \sqrt{2C_{NJ}(X)} \leq \sqrt{2J(X)}$ and $A_2(X) \leq 1 + \sqrt{J(X) - 1}$.

Useful results about above-mentioned constants are given in the following simple lemma.

Lemma 1.1. *Let $\|\ \|$ and $\|\ \|_1$ be equivalent norms on X, namely for $\alpha, \beta > 0$ $\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|$. Denote $X = (X, \|\ \|)$ and $X_1 = (X, \|\ \|_1)$. Then*

$$\frac{\alpha}{\beta}J(X) \leq J(X_1) \leq \frac{\beta}{\alpha}J(X),\ \frac{\alpha^2}{\beta^2}C_{NJ}(X) \leq C_{NJ}(X_1) \leq \frac{\beta^2}{\alpha^2}C_{NJ}(X)$$
$$\frac{\alpha^2}{\beta^2}E(X) \leq E(X_1) \leq \frac{\beta^2}{\alpha^2}E(X),\quad \frac{\alpha^2}{\beta^2}f(X) \leq f(X_1) \leq \frac{\beta^2}{\alpha^2}f(X)$$
$$\frac{\alpha}{\beta}A_2(X) \leq A_2(X_1) \leq \frac{\beta}{\alpha}A_2(X),\ \frac{\alpha^2}{\beta^2}C_Z(X) \leq C_Z(X_1) \leq \frac{\beta^2}{\alpha^2}C_Z(X).$$

If $\|x\| = \alpha\|x\|_1$, then $J(X) = J(X_1)$, $C_{NJ}(X) = C_{NJ}(X_1)$, $E(X) = E(X_1)$, $f(X) = f(X_1)$, $A_2(X) = A_2(X_1)$ and $C_Z(X) = C_Z(X_1)$.

Let $0 < p, q \leq \infty$. Then the best constant in the generalized complex Clarkson inequality

$$(|a+b|^q + |a-b|^q)^{1/q} \leq C(|a|^p + |b|^p)^{1/p}$$

is $C = C_{p,q}(\mathrm{C}) = \max\{2^{1-1/p}, 2^{1/q}, 2^{1/q-1/p+1/2}\}$.

Clarkson proved in 1936 that $C_{p,p'}(\mathrm{C}) = 2^{1/p'}$ for $1 \le p \le 2$, where $1/p+1/p' = 1$. For the remaining pairs of p and q the best constants were found by T.Koskela in 1979 and L.Maligranda and L.-E. Persson in 1992.

It is interesting when we instead of modulus of complex numbers have the norm of elements of Banach space X, $\|.\|$. For instance, when $1 < p \le 2$ we say that X satisfies the (p, p') Clarkson inequality (we write CI(p, p') holds in X) if

$$(\|x+y\|^{p'} + \|x-y\|^{p'})^{\frac{1}{p'}} \le 2^{\frac{1}{p'}}(\|x\|^p + \|y\|^p)^{\frac{1}{p}}$$

for every $x, y \in X$.

Let us note, that CI(p, p') holds in L_p for $1 < p \le 2$. Moreover

$$\|x+y\|_p^p + \|x-y\|_p^p \le 2(\|x\|_p^p + \|y\|_p^p)$$

for every $x, y \in L_p$.

For given $n \in \mathbf{N}$, $0 < p, q \le \infty$ and a Banach space X, we define *the n-th Khintchine constants* $K_{p,q}^n(X)$ to be the smallest of all numbers $C \ge 1$ such that

$$\left(\int_0^1 \|\sum_{k=1}^n r_k(t)x_k\|^q dt\right)^{\frac{1}{q}} \le C\left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}}$$

for every choice $x_1, \ldots, x_n \in X$, where $\{r_k\}_{k=1}^n$ are Rademacher functions. If $X = \mathbb{R}$ with the absolute value as the norm then we write shortly $K_{p,q}^n(\mathbb{R})$. Moreover, for $p = q$ we denote these constants by $t_{p,n}(X)$ as the numbers connected with the type p of the space X and if $p = q = 2$ we denote them shortly by $t_n(X)$ as the numbers connected with the type 2 of the space X. Note that $t_2(X) = \sqrt{C_{NJ}(X)}$. This can be explained by the fact, that if $0 < q < \infty$ we have equality

$$\int_0^1 \|\sum_{k=1}^n r_k(t)x_k\|^q dt = \frac{1}{2^n}\sum_{\theta_k=\pm1}\|\sum_{k=1}^n \theta_k x_k\|^q.$$

A Banach space X is of *type* p, $1 \le p \le 2$, if $T_{p,q}(X) := \sup_{n\in\mathbb{N}} K_{p,q}^n(X) < \infty$.

The case $q = p'$ is the most popular. (X is of type (p, p') if $T_{p,p'}(X) < \infty$ and this is called type constant.)

Pisier proved in 1973 that a Banach space X has non-trivial type, i.e., is of type p for some $p > 1$ if and only if it is B-convex.

Let $2 \le q \le +\infty$. We can say that X is of Rademacher cotype q if there exists $C \in \mathbf{R}$ such that

$$\left(\sum_{i=1}^n \|x_i\|^q\right)^{\frac{1}{q}} \le C\left(\int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|^{q'} dt\right)^{\frac{1}{q'}}$$

for every $x_1, x_2, ..., x_n \in X$. The smallest constant C satisfying this inequality is called the cotype constant of X.

The close relation between Clarkson type inequalities and the notion of type and cotype has been investigated in [31] and [32]. For example, for the case with type or cotype constant 1 we have the following theorem (see [32] and also [31], Theorems 3.2 and 3.4):

Theorem 1.2. *Let X be a Banach space and $1 < p \leq 2$. Then the following are equivalent:*
i) $C(p, p')$ holds in X.
ii) X is of Rademacher type p with type constant 1.
iii) X is of Rademacher cotype p' with cotype constant 1.
iv) $C(p, p')$ holds in X^*.
v) X^* is of Rademacher type p with type constant 1.
vi) X^* is of Rademacher cotype p' with cotype constant 1.

Here X^* as usual denotes the dual space of X.

1.2. **B-convexity and the n-th James constants.** Let us start with the notion of the uniformly non-l_1^n and B-convexity of a Banach space X. These notions are coming from James (1964) and Beck (1962).

For every natural number $n \geq 2$ we say, as in Giesy-James (1973), that a Banach space X is *uniformly non-l_n^1* if there exists an $\delta \in (0, 1)$ such that for every $x_1, \ldots, x_n \in B_X$ it holds $\| \sum_{k=1}^n \theta_k x_k \| \leq n(1 - \delta)$ for some choice of signs $\theta_1, \theta_2, \ldots, \theta_n$.

A Banach space X is called *B-convex* if it is uniformly non-l_n^1 for some $n \geq 2$.

In the connection to these two notions *the measure of uniformly non-l_n^1* or sometimes called *the measure of B-convexity* appeared. For given $n \in \mathbf{N}$ *the n-th James constants* $J_n(X)$ of a Banach space X are the numbers defined by

$$J_n(X) = \sup \left\{ \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\| : \; x_1, \ldots, x_n \in B_X \right\}.$$

Note that $J_1(X) = 1$, $J_2(X)$ is just the James constant $J(X)$ discussed in subsecection 1.1 and $J_n(l^1) = J_n(l_m^1) = n$ for $m \geq n$ by considering unit vectors.

It is clear that X is uniformly non-l_1^n if and only if $J_n(X) < n$, and X is B-convex if and only if $J_n(X) < n$ for some $n \geq 2$. Moreover, X is B-convex if and only if $\lim_{n \to \infty} \frac{J_n(X)}{n} = 0$.

The $n-$th James constants were studied by several authors e.g. Giesy in 1969, Pisier -1974, Woyczynski -1978, Kalton -1978, Kalton- Peck- Roberts -1984, Kadets- Kadets-1991, 1997 and Diestel-Jarchow-Tonge -1995. It seems that these constant appeared for the first time explicitely in a paper by Giesy in 1966 (for the references see [43]). Let us collect some properties of these constants:

(1) $1 \leq J_n(X) \leq n$*; if* $dimX = \infty$*, then* $J_n(X) \geq \sqrt{n}$*.*

(2) $J_n(X)$ *are increasing in* n *and* $J_{n+1}(X) \leq J_n(X) + 1$*.*

(3) $J_n(X)$ *is submultiplicative sequence, i.e.,* $J_{mn}(X) \leq J_m(X)J_m(X)$ *for all* $m, n \in \mathbb{N}$*.*

(4) If X *is a Hilbert space and* $\dim X \geq n$*, then* $J_n(X) = \sqrt{n}$*; The converse is not true in general.*

Before giving the proof of these properties in [43] we proved a useful result on n-James constants for finitely representable spaces. The notion of finitely representable spaces was introduced by James in 1972.

A Banach space X is said to be *finitely representable* in a Banach space Y if, for every $\varepsilon > 0$ and for every finite-dimensional subspace X_0 of X, there exists a subspace Y_0 of Y and an isomorphism T from X_0 onto Y_0 such that

$$\frac{1}{1+\varepsilon}\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\| \text{ for every } x \in X_0.$$

It is well-known that infinite dimensional Banach space X is B-convex if and only if l^1 is not finitely representable in X (see e.g. the books of J.Diestel, H.Jarshow, A.Tonge, 1995 or Kadetz and Kadetz, 1991.

Proposition 1.3. *If* X *is finitely representable in* Y*, then* $J_n(X) \leq J_n(Y)$ *for every* $n \geq 2$*.*

We used (in [43])this fact in the proof of (1). The results about the n-James constants for L^p spaces are in the next theorem.

Theorem 1.4. *If* $1 \leq p \leq \infty$*, then* $J_n(L^p(\mu)) \leq \max(n^{1/p}, n^{1-1/p})$*. Moreover, if* $1 \leq p \leq 2$ *and* $dimL^p(\mu) \geq n$*, then* $J_n(L^p(\mu)) = n^{1/p}$*; if* $2 < p \leq \infty$ *and* $dimL^p(\mu) = \infty$*, then* $\sqrt{n} \leq J_n(L^p(\mu)) \leq n^{1-1/p}$*.*

Proof. , let $1 \leq p \leq 2$. In this case we have Clarkson inequality

$$\|x+y\|_p^p + \|x-y\|_p^p \leq 2\left(\|x\|_p^p + \|y\|_p^p\right)$$

which is valid for all $x, y \in L^p(\mu)$. Then, by induction,

$$\sum_{\theta_k \pm 1} \|\sum_{k=1}^{n} \theta_k x_k\|_p^p \leq 2^n \sum_{k=1}^{n} \|x_k\|_p^p \text{ for all } x_1, \ldots, x_n \in X.$$

In fact, by the Clarkson inequality and the induction assumption (for $n-1$) we obtain

$$\sum_{\theta_k=\pm1} \|\sum_{k=1}^{n} \theta_k x_k\|_p^p = \sum_{\theta_k=\pm1} \|\sum_{k=1}^{n-1} \theta_k x_k + x_n\|_p^p + \sum_{\theta_k=\pm1} \|\sum_{k=1}^{n-1} \theta_k x_k - x_n\|_p^p$$

$$\leq 2 \sum_{\theta_k=\pm1} \left(\|\sum_{k=1}^{n-1} \theta_k x_k\|_p^p + \|x_n\|_p^p \right) \leq 2 \left(2^{n-1} \sum_{k=1}^{n-1} \|x_k\|_p^p + 2^{n-1}\|x_n\|_p^p \right) = 2^n \sum_{k=1}^{n} \|x_k\|_p^p,$$

and hence

$$\min_{\theta_k \pm 1} \|\sum_{k=1}^{n} \theta_k x_k\|_p^p \leq \frac{1}{2^n} \sum_{\theta_k=\pm1} \|\sum_{k=1}^{n} \theta_k x_k\|_p^p \leq \sum_{k=1}^{n} \|x_k\|_p^p.$$

This implies that if $x_1, x_2, \ldots, x_n \in B_{L^p}$, then $\min_{\theta_k \pm 1} \|\sum_{k=1}^{n} \theta_k x_k\|_p \leq n^{1/p}$, and hence $J_n(L^p) \leq n^{1/p}$.

Since $\dim L^p(\mu) \geq n$ we can find at least n pairwise disjoint subsets $A_1, \ldots, A_n$ of Ω such that $0 < \mu(A_k) < \infty$ for $k = 1, \ldots, n$. Define $x_k = \frac{\chi_{A_k}}{\mu(A_k)^{1/p}}$ for $k = 1, \ldots, n$. Then, for every choice of signs $\theta_1, \ldots, \theta_n$, we have that $\|\sum_{k=1}^{n} \theta_k x_k\|_p^p = n$ and, hence, $J_n(L^p(\mu)) \geq n^{1/p}$.

For $2 \leq p < \infty$ we use another Clarkson inequality

$$\|x+y\|_p^p + \|x-y\|_p^p \leq 2 \left(\|x\|_p^{p'} + \|y\|_p^{p'} \right)^{p-1},$$

from which we obtain two estimates

$$\min\left(\|x+y\|_p^p, \|x-y\|_p^p\right) \leq \left(\|x\|_p^{p'} + \|y\|_p^{p'} \right)^{p-1}$$

and

$$\min\left(\|x+y\|_p^{p'}, \|x-y\|_p^{p'}\right) \leq \|x\|_p^{p'} + \|y\|_p^{p'}.$$

These estimates give the required estimate for n-elements

$$\min_{\theta_k=\pm1} \|\sum_{k=1}^{n} \theta_k x_k\|_p^p \leq \left(\sum_{k=1}^{n} \|x_k\|_p^{p'} \right)^{p-1}.$$

In fact, by the first and second above estimates and the induction we obtain

$$\min_{\theta_k=\pm1} \|\sum_{k=1}^{n} \theta_k x_k\|_p^p = \min_{\theta_k=\pm1} \min \left(\|\sum_{k=1}^{n-1} \theta_k x_k + x_n\|_p^p, \|\sum_{k=1}^{n-1} \theta_k x_k - x_n\|_p^p \right)$$

$$\leq \min_{\theta_k=\pm1} \left(\|\sum_{k=1}^{n-1} \theta_k x_k\|_p^{p'} + \|x_n\|_p^{p'} \right)^{p-1}$$

$$= \min_{\theta_k=\pm1} \left[\min \left(\|\sum_{k=1}^{n-2} \theta_k x_k + x_{n-1}\|_p^{p'},\ \|\sum_{k=1}^{n-2} \theta_k x_k - x_{n-1}\|_p^{p'} + \|x_n\|_p^{p'} \right) \right]^{p-1}$$

$$\leq \left[\min_{\theta_k=\pm 1} \| \sum_{k=1}^{n-2} \theta_k x_k \|_p^{p'} + \|x_{n-1}\|_p^{p'} + \|x_n\|_p^{p'} \right]^{p-1} \leq \ldots \leq (\sum_{k=1}^{n} \|x_k\|_p^{p'})^{p-1}.$$

Hence, if $x_1, x_2, \ldots, x_n \in B_{L^p}$, then $\min_{\theta_k \pm 1} \| \sum_{k=1}^n \theta_k x_k \|_p \leq n^{1-1/p}$, and so $J_n(L^p) \leq n^{1-1/p}$.

The estimate from below $J_n(L^p(\mu)) \geq \sqrt{n}$ follows from the property (i). The proof is complete.

We can also consider the *n-th strong James constants* of a Banach space X defined by

$$J_n^s(X) := \sup \left\{ \min_{\theta_k=\pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\| : \ x_1, \ldots, x_n \in S_X \right\}.$$

Then obviously $J_n^s(X) \leq J_n(X)$ and $J_3^s(X) < J_3(X)$ for $X = l_2^\infty$, that is, when $X = \mathbb{R}^2$ with the norm of $x = (x_1, x_2)$ equal to $\|x\| = \max\{|x_1|, |x_2|\}$. In fact,

$$J_3^s(l_2^\infty) = 1 \text{ and } J_3(l_2^\infty) \geq 2.$$

The first equality can be proved by considering two extreme cases $x_1 = (1, a), x_2 = (1, b), x_3 = (1, c)$ and $x_1 = (1, a), x_2 = (1, b), x_3 = (c, 1)$ and the second estimate we obtain by taking $x_1 = (1, 1), x_2 = (-1, 1), x_3 = (0, \varepsilon)$ with $0 < \varepsilon < 1$ since then

$$\begin{aligned} J_3(l_2^\infty) &\geq \min\{\|x_1 + x_2 + x_3\|, \|x_1 + x_2 - x_3\|, \|x_1 - x_2 + x_3\|, \|x_1 - x_2 - x_3\|\} \\ &= \min\{2 + \varepsilon, 2 - \varepsilon, 2\} = 2 - \varepsilon, \end{aligned}$$

and our claim follows by letting $\varepsilon \to 0^+$.

We don't know any example of a Banach space X such that $\dim X \geq 3$ and $J_3^s(X) < J_3(X)$ but we guess that only $J_2^s(X) = J_2(X)$ for $\dim X \geq 2$ and the conjecture is that

$$J_n^s(X) < J_n(X) \text{ for } n \geq 3 \text{ and } \dim X \geq 3.$$

We easily see that $J_n^s(l^1) = J_n^s(l_m^1) = n$ for $m \geq n$ and in 2007 it was proved by L.Maligranda, N.Petrot and S.Suantai that for the Cesàro sequence spaces $ces_p, 1 < p \leq \infty$ we have equalities $J_n^s(ces_p) = n$ for all natural $n \geq 2$, which means that they are not B-convex.

Theorem 3 [43] estimates $J_n(X)$ by $J_n^s(X)$. I do not include here the proof, since it was done by L. Maligranda. For completness I mention, that as a corollary we get that for any Banach space X and $n \geq 2$ it yields that $J_n(X) < n$ if and only if $J_n^s(X) < n$, so in the definition of uniformly non-l_n^1 space we can have elements from the unit sphere or from the unit ball (mentioned also by Kaminska-Turett in 1987).

1.3. **Type and the n-th Khintchine constants of Banach spaces.** The Khintchine constants $K^n_{p,q}(X)$ for some special choices of p, q were studied by several authors, e.g. Pisier in 1973, Enflo-Lindenstrauss-Pisier in 1975 and Figiel-Lindenstrauss-Milman in 1977 considered $t_n(X)$, Maurey-Pisier in 1976, Woyczyński in 1978, and Milman-Schechtman in 1986 investigated $t_{p,n}(X)$ for $1 \leq p \leq 2$. The following properties for the n-th Khintchine constants hold (details in[43] and in [54] in notations $T^n_{p,s}$):

(1) $1 \leq K^n_{p,q}(\mathbb{R}) \leq K^n_{p,q}(X) \leq n^{(1-1/p)_+}$.

(2) $K^n_{p,q}(X)$ are increasing in n, p, q, $K^n_{1,1}(X) = 1$ and $K^n_{p,q}(X) \leq n^{\frac{1}{r}-\frac{1}{p}} K^n_{r,q}(X)$ for all $0 < q \leq \infty, 0 < r \leq p \leq \infty$.

(3) If $1 \leq q < \infty$, then $K^n_{p,q}(X)$ are subadditive in n, that is, $K^{m+n}_{p,q}(X) \leq K^m_{p,q}(X) + K^n_{p,q}(X)$ for all $m, n \in \mathbb{N}$.

(4) If $0 < p \leq q < \infty$, then $K^n_{p,q}(X)$ are submultiplicative in n, that is, $K^{mn}_{p,q}(X) \leq K^m_{p,q}(X) K^n_{p,q}(X)$ for all $m, n \in \mathbb{N}$.

(5) If X is a Hilbert space, then $K^n_{p,q}(X) = 1$ for $0 < p \leq 2 \leq q \leq \infty$; $K^n_{2,2}(X) = 1$ for $n \geq 2$ if and only if X is a Hilbert space.

(6) If $1 \leq r < \infty$ and $0 < p \leq r \leq q < \infty$, then $K^n_{p,q}(L^r(\mu)) = K^n_{p,q}(\mathbb{R})$. In particular, if $1 \leq r \leq \infty$ and $0 < p \leq r \leq q \leq 2$, then $K^n_{p,q}(L^r(\mu)) = 1$.

The following result is an estimate of the n-th Khintchine constants by the n-th James constants. It is written as it is in [43], but it was proved in a slightly different form still in [54]. For the proof we need the following crucial lemma , which in fact has been proved in Kutzarova-Nikolova-Zachariades - Lemma 6 in [41].

Lemma 1.5. *Let X be a normed space and $n \geq 2$. Then, for every $x_1, x_2, \ldots, x_n \in X$, with $\|x_n\| \leq \|x_k\|$ for $k = 1, 2, \ldots, n-1$, there exist $\theta_1, \ldots, \theta_n \in \{-1, 1\}$ such that*

$$\|\sum_{k=1}^{n} \theta_k x_k\| \leq \sum_{k=1}^{n-1} \|x_k\| + [J^s_n(X) - n + 1]\, \|x_n\|.$$

Proof. If $x_n = 0$ the lemma is clear. Let $x_n \neq 0$. Then there exists a choice of signs $\theta_1, \ldots, \theta_n$ such that $\| \sum_{k=1}^{n} \theta_k \frac{x_k}{\|x_k\|} \| \leq J^s_n(X)$ and so we have

$$\|\sum_{k=1}^{n} \theta_k x_k\| = \|\sum_{k=1}^{n} \theta_k \left(1 - \frac{\|x_n\|}{\|x_k\|}\right) x_k + \sum_{k=1}^{n} \theta_k \frac{\|x_n\|}{\|x_k\|} x_k\|$$

$$\leq \sum_{k=1}^{n}\left(1-\frac{\|x_n\|}{\|x_k\|}\right)\|x_k\|+\|x_n\|\|\sum_{k=1}^{n}\theta_k\frac{x_k}{\|x_k\|}\|\leq\sum_{k=1}^{n-1}\|x_k\|+\left[J_n^s(X)-n+1\right]\|x_n\|.$$

Theorem 1.6. *Let X be a Banach space X and $1\leq p,q\leq\infty$, $n\geq 2$. Then*

$$\frac{J_n(X)}{n^{\frac{1}{p}}}\leq K_{p,q}^n(X)\leq\frac{1}{2^{\frac{n-1}{q}}}\left[2^{\frac{(n-1)p'}{q}}(n-1)+c_n^{\frac{p'}{q}}\right]^{\frac{1}{p'}},$$

where $c_n=a_n^q+2^{n-1}-1$ and $a_n=\left[J_n^s(X)-n+1\right]_+$. For $p=1$ the right hand side is as usual interpreted is equal to 1.

Proof. As usual, $\frac{1}{p}+\frac{1}{p'}=1$. It is clear that

$$\min_{\theta_k=\pm1}\|\sum_{k=1}^{n}\theta_kx_k\|\leq\left(\frac{1}{2^n}\sum_{\theta_k=\pm1}\|\sum\theta_kx_k\|^q\right)^{\frac{1}{q}}\leq K_{p,q}^n(X)(\sum_{k=1}^{n}\|x_k\|^p)^{\frac{1}{p}}\leq K_{p,q}^n(X)\,n^{\frac{1}{p}},$$

which gives the left hand side inequality. It is also clear that $J_n(X)\leq K_{\infty,q}^n(X)$.

To prove the upper estimate we may suppose, without loss of generality, that $x_1,\ldots,x_n\in X$ are not all zero, $\|x_n\|\leq\|x_k\|$ for every $k=2,3,\ldots,n-1$ and $\sum_{k=1}^{n-1}\|x_k\|=1$.

Let $1<p<\infty$ and $1\leq q<\infty$. By using Lemma and Minkowski's inequality, we obtain that

$$\frac{1}{2^{\frac{1}{q}}}\left[\sum_{\theta_k=\pm1}\|\sum\theta_kx_k\|^q\right]^{\frac{1}{q}}\leq\left[\left(\sum_{k=1}^{n-1}\|x_k\|+a_n\|x_n\|\right)^q+(2^{n-1}-1)\left(\sum_{k=1}^{n-1}\|x_k\|+\|x_n\|\right)^q\right]^{\frac{1}{q}}$$

$$\leq\left[2^{n-1}\left(\sum_{k=1}^{n-1}\|x_k\|\right)^q\right]^{\frac{1}{q}}+\left[a_n^q\|x_n\|^q+(2^{n-1}-1)\|x_n\|^q\right]^{\frac{1}{q}}=2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}\|x_n\|.$$

Hence

$$K_{p,q}^n(X)\leq\frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}\|x_n\|}{2^{\frac{n-1}{q}}(\sum_{k=1}^{n}\|x_k\|^p)^{\frac{1}{p}}}.$$

Moreover, according to Hölder's inequality,

$$1=\sum_{k=1}^{n-1}\|x_k\|\leq(n-1)^{\frac{1}{p'}}(\sum_{k=1}^{n-1}\|x_k\|^p)^{\frac{1}{p}},$$

and, thus, $\sum_{k=1}^{n-1}\|x_k\|^p\geq(n-1)^{1-p}$. We get $K_{p,q}^n(X)\leq\frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}\|x_n\|}{2^{\frac{n-1}{q}}(\|x_n\|^p+(n-1)^{1-p})^{\frac{1}{p}}}$. Let us consider the function $f(t)=\frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}t}{2^{\frac{n-1}{q}}(t^p+(n-1)^{1-p})^{\frac{1}{p}}}$ for $t\geq0$. It is easy to see that

$f(t) \le f(t_0)$, where $t_0 = \frac{c_n^{\frac{1}{q(p-1)}}}{(n-1)2^{\frac{n-1}{q(p-1)}}}$. Thus we obtain that

$$K_{p,q}^n(X) \le \frac{1}{2^{\frac{n-1}{q}}} \left[2^{\frac{(n-1)p'}{q}}(n-1) + c_n^{\frac{p'}{q}} \right]^{\frac{1}{p'}}.$$

Therefore the right hand side of desired inequality holds.

If $p = 1$, then the estimate $K_{1,q}^n(X) \le 1$ is clear and the right hand side is equal to

$$2^{(1-n)/q} \max\{2^{(n-1)/q}, \ldots, 2^{(n-1)/q}, c_n^{1/q}\} = 1.$$

If $p = \infty$, then $p' = 1$ and $1 = \sum_{k=1}^{n-1} \|x_k\| \le (n-1) \max_{k=1,\ldots,n-1} \|x_k\|$ from which it follows that $\max_{k=1,\ldots,n} \|x_k\| = \max_{k=1,\ldots,n-1} \|x_k\| \ge \frac{1}{n-1}$ and, thus,

$$K_{\infty,q}^n(X) \le \frac{2^{\frac{n-1}{q}} + c_n^{\frac{1}{q}} \|x_n\|}{2^{\frac{n-1}{q}} \max_{k=1,\ldots,n} \|x_k\|} \le \frac{2^{\frac{n-1}{q}} + c_n^{\frac{1}{q}} \frac{1}{n-1}}{2^{\frac{n-1}{q}} \frac{1}{n-1}}.$$

This gives the required estimate $K_{\infty,q}^n(X) \le n - 1 + \frac{c_n^{\frac{1}{q}}}{2^{\frac{n-1}{q}}}$.

If $q = \infty$, then $\max_{\theta_k = \pm 1} \| \sum_{k=1}^n \theta_k x_k \| \le \sum_{k=1}^{n-1} \|x_k\| + \|x_n\| = 1 + t$, and

$$K_{p,\infty}^n(X) \le \frac{1+t}{(\sum_{k=1}^n \|x_k\|^p)^{1/p}} \le \frac{1+t}{[t^p + (n-1)]^{1/p}} := g(t).$$

The function g has maximum at $t_0 = \frac{1}{n-1}$ and, hence, $g(t) \le g(t_0) = n^{1/p'}$, which is again the right hand side and the proof is complete.

Corollary 1.7. *If $n \ge 2$, then*

$$\frac{J_n(X)}{\sqrt{n}} \le t_n(X) = K_{2,2}^n(X) \le 2^{\frac{1-n}{2}} \left\{ 2^{n-1}n - 1 + [J_n^s(X) - n + 1]_+^2 \right\}.$$

In particular, $\frac{J(X)^2}{2} \le C_{NJ}(X) = K_{2,2}^2(X)^2 \le \frac{J(X)^2}{2} + 2 - J(X)$.

Corollary 1.8. *An infinite dimensional Banach space X is uniformly non-l_n^1 if and only if $K_{p,q}^n(X) < n^{1/p'}$ for $1 < p < \infty, 1 \le q < \infty$.*

(Pisier in 1973 proved this for the case $p = q = 2$.)

Corollary 1.9. *Let X be a Banach space. For $n \ge 2$ and $1 < p < \infty, 1 \le q < \infty$ fixed we have that $J_n(X) = n$ if and only if $K_{p,q}^n(X) = n^{1/p'}$.*

The assertion of the following proposition is known for $p = 1$ and means that the Banach space X is B-convex (D.P.Giesy, R.C.James in 1973). We extend this result to $1 \le p < 2$.

Proposition 1.10. *Let X be an infinite dimensional Banach space and $1 \le p < 2$. The following conditions are equivalent*

(i) X is of type strictly bigger than p.

(ii) $\lim_{n\to\infty} \frac{J_n(X)}{n^{1/p}} = 0$.

(iii) $\inf_{n\geq 2} \frac{J_n(X)}{n^{1/p}} < 1$.
(iv) l^p *is not finitely representable in* X.

Proof. (i) $\Rightarrow$ (ii). Let X be of type r for some $p < r \leq 2$. Then, according to the first estimate in (7), we obtain that $\sup_{n\geq 2} \frac{J_n(X)}{n^{\frac{1}{r}}} < \infty$. Hence, $\lim_{n\to\infty} \frac{J_n(X)}{n^{\frac{1}{p}}} = \lim_{n\to\infty}[\frac{J_n(X)}{n^{\frac{1}{r}}} n^{\frac{1}{r}-\frac{1}{p}}] = 0$.
(ii) $\Rightarrow$ (iii). This implication is obvious.
(iii) $\Rightarrow$ (iv). Since $J_n(l^p) = n^{1/p}$ we conclude that l^p is not finitely representable in X. In fact, if it is so, then by Proposition 1 we will have that $n^{1/p} = J_n(l^{p)} \leq J_n(X)$, which contradicts the assumption (iii).
(iv) $\Rightarrow$ (i). We use the Maurey-Pisier theorem : If X is an infinite dimensional Banach space, then $l^{p(X)}$ is finitely representable in X and even more: l^r is finitely representable in X for every $r \in [p(X), 2]$, where $p(X) := \sup\{p \geq 1 : X \textit{ is of type } p\}$.

From this fact we conclude that $p < p(X)$ and, thus, X is of type strictly bigger than p. The proof is complete.

Corollary 1.11. *If* X *be an infinite dimensional Banach space,* $1 < p \leq 2$ *and* $\sup_{n\geq 2} \frac{J_n(X)}{n^{1/p}} < \infty$, *then* X *is of type* r *for every* $1 < r < p$.

Proof. The result follows from Proposition 3 since for $1 < r < p$ we have that $\lim_{n\to\infty} \frac{J_n(X)}{n^{1/r}} = \lim_{n\to\infty} \frac{J_n(X)}{n^{1/p}} n^{1/p-1/r} = 0$.

Theorem 1.12. *If* X *is an infinite dimensional Banach space, then*

$$\lim_{n\to\infty} \frac{\ln n}{\ln J_n(X)} = \sup_{n\geq 2} \frac{\ln n}{\ln J_n(X)} = p(X).$$

Proof. The first equality in the assertion of the theorem follows from the property of submultiplicative sequences. In fact, if $\{a_n\}$ is a submultiplicative sequence, then $\lim_{n\to\infty} \frac{\ln a_n}{\ln n}$ exists and is equal to $\inf_{n\geq 2} \frac{\ln a_n}{\ln n}$. Define $l(X) := \sup_{n\geq 2} \frac{\ln n}{\ln J_n(X)}$ and note also that from the estimates $\sqrt{n} \leq J_n(X) \leq n$ we obtain that $1 \leq \frac{\ln n}{\ln J_n(X)} \leq 2$ and, hence, $1 \leq l(X) \leq 2$.

Now, we suppose that $p(X) < l(X)$. Let $p(X) < r < l(X)$. Then there exists $m \geq 2$ such that $r < \frac{\ln m}{\ln J_m(X)}$. We conclude that $\frac{J_m(X)}{m^{\frac{1}{r}}} < 1$ and, thus, according to Proposition 3, X is of type r which is a contradiction.
We suppose that $l(X) < p(X)$. Let $l(X) < s < p(X)$. Then we have that $1 < \frac{J_n(X)}{n^{\frac{1}{s}}}$ for every $n \geq 2$. Thus, again by using Proposition 3, we find that X

is not of type bigger than s which is a contradiction. Thus $l(X) = p(X)$ and the proof is complete.

Note that Woyczyński in 1978 proved the following related result: if X is an infinite dimensional Banach space and $0 < p < \infty$, then

$$p(X) = \lim_{n\to\infty} \frac{\ln n}{\ln[n^{1/p} K^n_{p,p}(X)]},$$

and Milman-Schechtman in 1986 proved for $p(X) \le p \le 2$ that

$$p(X) = \lim_{n\to\infty} \frac{\ln n}{\ln[n^{1/p} K^n_{p,2}(X)]}$$

or equivalently

$$\lim_{n\to\infty} \frac{\ln K^n_{p,2}(X)}{\ln n} = \frac{1}{p(X)} - \frac{1}{p}.$$

The idea and the proof of the next proposition is given by L.Maligranda. We include it here for the completness.

Proposition. If $p \ge 2$, then

$$J_n(L^p(\mu)) \le \min \left[n^{1-1/p}, \left(\int_0^1 |\sum_{k=1}^n r_k(t)|^p \ dt \right)^{1/p} \right].$$

First estimate was already proved in Theorem 1.4. The other estimate we are getting from Theorem 1.6 and the Figiel-Iwaniec-Pełczyński (1984) estimate

$$J_n(L^p(\mu)) \le n^{1/p} K^n_{p,p}(L^p(\mu)) = K^n_{p,p}(\mathbb{R}) \le \left(\int_0^1 |\sum_{k=1}^n r_k(t)|^p dt \right)^{1/p}.$$

Note that for $n = 2$ and $n = 3$ the constant $n^{1-1/p}$ is smaller than $(\int_0^1 |\sum_{k=1}^n r_k(t)|^p dt)^{1/p}$ and for large n we have reverse inequality. Moreover, $\lim_{n\to\infty} \frac{J_n(L^p)}{n} = \frac{1}{2}$.

Example. For $1 \le p \le 2$ and $\lambda \ge 1$ let $X_{\lambda,p}$ be the space $L^p[0,1]$ with the norm

$$\|x\|_{\lambda,p} = \max\{\|x\|_p, \lambda \|x\|_1\}.$$

Then

$$J_n(X_{\lambda,p}) = \min\{n, \lambda n^{1/p}\}, \text{ and } K^n_{p,p}(X_{\lambda,p}) = \min\{n^{1-1/p}, \lambda\}.$$

Note, that if $\alpha\|x\| \le \|x\|_1 \le \beta\|x\|$ and $X = (X, || \, ||)$ and $X_1 = (X, || \, ||_1)$, then

$$\frac{\alpha}{\beta} J_n(X) \le J_n(X_1) \le \frac{\beta}{\alpha} J_n(X)$$

and the similar inequality is true if J_n is replaced by $K^n_{p,q}$. Then inequalities from above follow from the estimates $\|x\|_p \le \|x\|_{\lambda,p} \le \lambda\|x\|_p$ for all $x \in L^p$. Equalities we get by taking functions $x_{k,n} = a\chi_{[\frac{k-1}{n},\frac{k}{n})}$ for $k = 1, 2, \ldots, n, n = 2, 3, \ldots$ with $a = \min(n^{1/p}, n/\lambda)$, since

$$\|x_{k,n}\|_{\lambda,p} = \max\{\frac{a}{n^{1/p}}, \lambda\frac{a}{n}\} = \frac{a}{\min(n^{1/p}, n/\lambda)} = 1$$

and

$$\sum_{\theta_k=\pm1} \|\sum_{k=1}^{n} \theta_k x_{k,n}\|_{\lambda,p} = \max(a, \lambda a) = \lambda a.$$

Thus $J_n(X_{\lambda,p}) \geq \lambda a$ and $K^n_{p,p}(X_{\lambda,p}) \geq \frac{a\lambda}{n^{1/p}} = \min\{\lambda, n^{1-1/p}\}$, which means we have estimates from below, and consequenly equalities.

1.4. **Constants for Banach spaces with absolute normalized norms which corresponding functions are comparable.** Here we consider space $\mathbf{C}^2$ equipped with an absolute normalized norm $\|.\|$ i.e. such that

$$\|(x, y)\| = \|(|x|, |y|)\| \text{ and } \|(1, 0)\| = \|(0, 1)\| = 1.$$

Let the set N_a be a family of all absolute and normalized norms on $\mathbf{C}^2$ and let Ψ_2 denotes the family of all convex functions ψ on $[0, 1]$ with $\psi(0) = \psi(1) = 1$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $0 \leq t \leq 1$. It is known that Ψ_2 is in one-to-one correspondence with N_a. Namely, Bonsall and Duncan in 1973 showed that if $\|.\| \in N_a$ then $\psi(t) = \|(1 - t, t)\| \in \Psi_2$ and conversely, if $\psi \in \Psi_2$, then

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\frac{|y|}{|x|+|y|}\right), & (x, y) \neq (0, 0); \\ 0 & (x, y) = (0, 0) \end{cases}$$

is a norm and $\|.\|_\psi \in N_a$.

Some examples of such norms and the corresponding functions are the following:

1. The convex functions $\psi_p(t) = [(1 - t)^p + t^p]^{1/p},\ 1 \leq p < \infty$, correspond to l_p-norms $\|.\|_p$. For $p = \infty$, the function $\psi_\infty(t) = \max\{t, 1 - t\}$ corresponds to a norm $\|.\|_\infty$.

2. Let

$$\|(x, y)\|_{\omega,q} = (x^{*q} + \omega y^{*q})^{\frac{1}{q}}$$

where $(x*, y*)$ is a non-increasing rearrangement of $(|x|, |y|)$. If $0 < \omega < 1,\ 1 \leq q < \infty$, then $\|.\|_{\omega,q}$ is a norm of two-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ and it belongs to N_a. The corresponding function from Ψ_2 is

$$\psi_{\omega,q}(t) = \begin{cases} ((1 - t)^q + \omega t^q)^{\frac{1}{q}}, & \text{if } 0 \leq t \leq \frac{1}{2}; \\ (t^q + \omega(1 - t)^q)^{\frac{1}{q}}, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

3. Let $1 \leq q < p \leq \infty$ and $2^{1/p-1/q} < \lambda < 1$. Let $\psi_{p,q,\lambda} = \max\{\psi_p, \lambda\psi_q\}$. Then the corresponding norm is $\|.\|_{p,q,\lambda} = \max\{\|.\|_p, \lambda\|.\|_q\}$.

4. Let $1 \leq q < p \leq \infty$, $r > 0$ and $\alpha > 0$. Let

$$\psi_{\alpha,p,q,r}(t) = (1+\alpha)^{-\frac{1}{r}}[((1-t)^q + t^q)^{\frac{r}{q}} + \alpha((1-t)^p + t^p)^{\frac{r}{p}}]^{\frac{1}{r}}.$$

Then the corresponding norm is $\|.\|_{\alpha,p,q,r} = (1+\alpha)^{-\frac{1}{r}}(\|.\|_q^r + \alpha\|.\|_p^r)^{\frac{1}{r}}$. If $p = \infty$, the function is

$$(1+\alpha)^{-\frac{1}{r}}[((1-t)^q + t^q)^{\frac{r}{q}} + \alpha(\max\{(1-t), t\})^r]^{\frac{1}{r}}.$$

Then the corresponding norm is $\|.\|_{\alpha,\infty,q,r} = (1+\alpha)^{-\frac{1}{r}}(\|.\|_q^r + \alpha\|.\|_\infty^r)^{\frac{1}{r}}$.

5. The Cesàro sequence space is very useful in the theory of matrix operators and others. Here we consider a two-dimensional Cesàro space $ces_p^{(2)}$ $(p \geq 1)$, which is just $\mathbf{R}^2$ equipped with the norm defined by

$$\|(x,y)\|_{ces_p^{(2)}} = \left(|x|^p + \left(\frac{|x|+|y|}{2}\right)^p\right)^{\frac{1}{p}}.$$

It is not an absolute normalized norm, but following [65] consider the norm

$$|(x,y)| = \left\|\left(\frac{2x}{(1+2^p)^{1/p}}, 2y\right)\right\|_{ces_p^{(2)}}.$$

It follows that $ces_p^{(2)}$ is isometrically isomorphic to $(\mathbf{R}^2, |.|)$ and $|.|$ is an absolute normalized norm. The corresponding function is

$$\psi_{ces}(t) = \left(\frac{2^p(1-t)^p}{1+2^p} + \left(\frac{1-t}{(1+2^p)^{1/p}} + t\right)^p\right)^{1/p}.$$

This norm $|.|$ is not symmetric.

Consider the space with the norm

$$\|(x,y)\|_{*ces_p} = \frac{2}{(1+2^p)^{1/p}}\|(x^*,y^*)\|_{ces_p^{(2)}}$$

where (x^*, y^*) is a non-increasing rearrangement of $(|x|, |y|)$. In fact this is the norm $\|(x,y)\|_{*ces_p} = (1+2^p)^{-1/p}(2^p\|(x,y)\|_\infty^p + \|(x,y)\|_1^p)^{1/p}$. This norm is absolute, normalized and symmetric. The above-mentioned norm can be generalized using substitution $\frac{1}{2^p} \to \omega$. So, consider the space with the norm

$$\|(x,y\|_{\omega,*,ces_p} = \left(\frac{x^{*p}}{1+\omega} + \frac{\omega(x^*+y^*)^p}{1+\omega}\right)^{\frac{1}{p}} = \frac{1}{(1+\omega)^{\frac{1}{p}}}(\|(x,y)\|_\infty^p + \omega\|(x,y)\|_1^p)^{\frac{1}{p}}.$$

This norm is absolute, normalized and symmetric. The corresponding convex function is

$$\psi_{\omega,*,ces_p}(t) = \begin{cases} \left(\frac{(1-t)^p}{1+\omega} + \frac{\omega}{1+\omega}\right)^{1/p}, & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \left(\frac{t^p}{1+\omega} + \frac{\omega}{1+\omega}\right)^{1/p}, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

In the paper [44] Mitani and Saito characterized the James constant of an absolute norm on $\mathbf{C}^2$. In the following text we will use notation X_ψ for the space with norm $\|.\|_\psi$ and write $J(\|.\|_\psi)$ instead $J(X_\psi)$. When $\psi = \psi_p$ we write $J(\|.\|_p)$. They gave the following theorem:

Theorem 1.13. *Let $\psi \in \Psi_2$. If ψ is symmetric with respect to $t = \frac{1}{2}$, then*

$$J(\|.\|_\psi) = \max_{0\le t\le 1/2} \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right).$$

Throughout this subsection we will use notation f_ψ for the function on the right-hand side of the above equality, i.e.

$$f_\psi(t) = \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right).$$

As a consequence of the above theorem, Mitani and Saito obtained result for function ψ which is comparable with ψ_2.

In this subsection subjects of investigation are two norms $\|.\|_\psi$ and $\|.\|_{\psi_*}$ with comparable corresponding functions ψ and ψ_*. We obtain theorems which give us relations between considered constants in the case when $\psi \le \psi_*$ or $\psi \ge \psi_*$ and generalize resultsof Mitani-Saito's and Cui-Wang's. Our results intersect with results from [71] where James type constants are considered, see Remark 1.19. In our paper [60] results for means of two norms are given. Here we will give only the application of our results for the norms from Examples 3.4.5. and some other concrete norms .

Consider functions $\psi, \psi_* \in \Psi_2$ with property $\psi \ge \psi_*$. We proved in [60] the following

Theorem 1.14. *Let $\psi, \psi_* \in \Psi_2, \psi \ge \psi_*$, and let us denote $M_1 = \max_{t\in[0,1]} \frac{\psi(t)}{\psi_*(t)}$.*
a) Then

$$J(\|.\|_\psi) \le M_1 J(\|.\|_{\psi_*}),\ C_{NJ}(\|.\|_\psi) \le M_1^2 C_{NJ}(\|.\|_{\psi_*}),\ E(\|.\|_\psi) \le M_1^2 E(\|.\|_{\psi_*}),$$
$$f(\|.\|_\psi) \ge \frac{f(\|.\|_{\psi_*})}{M_1^2},\ A_2(\|.\|_\psi) \le M_1 A_2(\|.\|_{\psi_*}),\ C_Z(\|.\|_\psi) \le M_1^2 C_Z(\|.\|_{\psi_*}).$$

b) Let, in addition, ψ be symmetric with respect to $\frac{1}{2}$.
If $M_1 J(\|.\|_{\psi_*}) \in f_\psi([0, \frac{1}{2}])$, then $J(\|.\|_\psi) = M_1 J(\|.\|_{\psi_*})$ holds.
If $\sqrt{2} M_1 \sqrt{C_{NJ}(\|.\|_{\psi_*})} \in f_\psi([0, \frac{1}{2}])$, then $C_{NJ}(\|.\|_\psi) = M_1^2 C_{NJ}(\|.\|_{\psi_*})$ holds.
If $\frac{1}{\sqrt{2}} M_1 \sqrt{E(\|.\|_{\psi_*})} \in f_\psi([0, \frac{1}{2}])$, then $E(\|.\|_\psi) = M_1^2 E(\|.\|_{\psi_*})$ holds.
If $\frac{\sqrt{f(\|.\|_{\psi_*})}}{\sqrt{2} M_1} \in f_\psi([0, \frac{1}{2}])$, then $f(\|.\|_\psi) = \frac{f(\|.\|_{\psi_*})}{M_1^2}$ holds.
If $M_1 A_2(\|.\|_{\psi_*}) \in f_\psi([0, \frac{1}{2}])$, then $A_2(\|.\|_\psi) = M_1 A_2(\|.\|_{\psi_*})$ holds.

If $\sqrt{2}M_1\sqrt{C_Z(\|.\|_{\psi_*})} \in f_\psi([0,\frac{1}{2}])$, *then* $C_Z(\|.\|_\psi) = M_1^2 C_Z(\|.\|_{\psi_*})$ *holds.*

As a consequence of the above theorem a theorem is proved [60], saying that if the function $\frac{\psi}{\psi_*}$ attains its maximum at 1/2, then in some sense the formula for the corresponding constant for the function ψ_* has the same form for the function ψ_*, for instance, if J$(\|.\|_{\psi_*}) = 2\psi_*(\frac{1}{2})$, then $J(\|.\|_\psi) = 2\psi(\frac{1}{2})$. We will not give here the formulas for the other constants, mentioned above and the proof of such formulas, but we will give just the consequence of such a theorem, namely if the function ψ_* is the corresponding function for ℓ_p-norm, then we obtain the following result.

Corollary 1.15. *Let* $\psi(t) \in \Psi_2, \psi \geq \psi_p, (1 \leq p \leq 2)$ *and the function* ψ/ψ_p *attains its maximum at* $t = \frac{1}{2}$. *Then*

$$J(\|.\|_\psi) = A_2(\|.\|_\psi) = 2\psi(\frac{1}{2}), \quad E(\|.\|_\psi) = 8\psi^2(\frac{1}{2}),$$

$$C_{NJ}(\|.\|_\psi) = C_Z(\|.\|_\psi) = 2\psi^2(\frac{1}{2}), \quad f(\|.\|_\psi) = \frac{2}{\psi^2(\frac{1}{2})}.$$

The proof is based on the above mentioned theorem and on the known formulas for the constant in the case of l_p- norm, For instance, it is known that $J(\|.\|_p) = 2^{1/p}$, which is equal to $2\psi_p(\frac{1}{2})$, and hence $J(\|.\|_\psi) = 2\psi(\frac{1}{2})$.

Using the same idea we obtain corresponding results for the function ψ which are smaller than ψ_*. Let us formulate only the corresponding result (one can find the theorems in the paper [60].

Corollary 1.16. *Let* $\psi(t) \in \Psi_2, \psi \leq \psi_p, (2 \leq p)$ *and the function* ψ_p/ψ *attains its maximum at* $t = \frac{1}{2}$. *Then*

$$J(\|.\|_\psi) = A_2(\|.\|_\psi) = \frac{1}{\psi(\frac{1}{2})}, \quad E(\|.\|_\psi) = \frac{2}{\psi^2(\frac{1}{2})},$$

$$C_{NJ}(\|.\|_\psi) = C_Z(\|.\|_\psi) = \frac{1}{2\psi^2(\frac{1}{2})}, \quad f(\|.\|_\psi) = 8\psi^2(\frac{1}{2}).$$

Remark 1.17. If $p = 2$, then the results from Corollaries 1.15 and 1.16 for $J(\|.\|_\psi)$ and $C_{NJ}(\|.\|_\psi)$ are given in [44]. If $1 \leq p < \infty$, then results for $E(\|.\|_\psi)$, $f(\|.\|_\psi)$ are obtained in [20].

Remark 1.18. The functions ψ which satisfy assumptions of Corollaries 1.15 and 1.16 generate norms for which the constants C_{NJ} and C_Z coincide, i.e. for which Zbăganu's conjecture is true. Moreover, equality between constants J and A_2 holds for these norms.

Remark 1.19. Very recently some results about the general James type constant $J_{X,p}(t) = \sup\{\left(\frac{\|x+ty\|^p+\|x-ty\|^p}{2}\right)^{\frac{1}{p}}, x, y \in S_X\}$ appear. Namely for $J_{X,p}(1)$ in [71] for $p \geq 1$ is calculated if ψ is comparable with ψ_r. Let us note that $J_{X,1}(1) = A_2(X)$ and $J_{X,2}(1) = \left(\frac{E(X)}{2}\right)^{\frac{1}{2}} = (2C'_{NJ}(X))^{\frac{1}{2}}$. Results from our Corollaries 1.15 and 1.16 coincide with the result for $J_{X,1}(1)$ from [71]. Note that for $J_{X,2}(1)$ we get result also in the case $\psi \geq \psi_2$, so this can be regarded as a complement to Theorem 2.5 from [71]. For $\psi \leq \psi_r$ our result is the same as the result which follows from Theorem 2.4 in the case $p = 2$. Note that for $E(X)(= (2J_{X,2}(1))^2)$ results from our Corollaries were obtained also in [20]. As it is mentioned by Takahashi (2007, Yokohama), $J_{X,-\infty}(1) = J(X)$ so the results from [71] can be continued also for $p = -\infty$ - see our Corollaries 1.15 and 1.16.

The next theorem we present with a proof, which is simple and can show to the reader ideas of the proof of mentioned above theorems.

Theorem 1.20. *Let $\psi \in \Psi_2$ and let $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. Let us denote $M_1 = \max_{t\in[0,\frac{1}{2}]} \frac{\psi(t)}{\psi_2(t)}$, $M_2 = \max_{t\in[0,\frac{1}{2}]} \frac{\psi_2(t)}{\psi(t)}$.*

If M_1 is attained at $t = \frac{1}{2}$, then $E(\|.\|_\psi) = 4M_1^2M_2^2$.

If M_2 is attained at $t = \frac{1}{2}$, then $f(\|.\|_\psi) = \frac{4}{M_1^2M_2^2}$.

If at least one of those maximums is attained at $t = \frac{1}{2}$, then $C_{NJ}(\|.\|_\psi) = M_1^2M_2^2$.

Proof. Suppose $M_1 = \frac{\psi(\frac{1}{2})}{\psi_2(\frac{1}{2})}$. For arbitrary $t \in \langle 0,1\rangle$, put $x_0 = \frac{1}{\psi(t)}(t, 1-t)$, $y = \frac{1}{\psi(t)}(1-t, t)$. Then $\|x_0\|_\psi^2 = \|y_0\|_\psi^2 = 1$, $\|x_0+y_0\|_\psi^2 = \frac{4\psi^2(1/2)}{\psi^2(t)}$ and $\|x_0-y_0\|_\psi^2 = \frac{1}{\psi^2(t)}\|(2t-1, 1-2t)\|_\psi^2 = \frac{4(2t-1)^2\psi(\frac{1}{2})^2}{\psi^2(t)}$. Then

$$\begin{aligned} \|x_0+y_0\|_\psi^2 + \|x_0-y_0\|_\psi^2 &= \frac{4[(2t-1)^2+1]\psi^2(\frac{1}{2})}{\psi^2(t)} \\ &= \frac{4[(1-t)^2+t^2]\psi^2(\frac{1}{2})}{\psi^2(t)/2} = \frac{4\psi^2(\frac{1}{2})\psi_2^2(t)}{\psi_2^2(\frac{1}{2})\psi^2(t)} = M_1^2\frac{4\psi_2^2(t)}{\psi^2(t)}. \end{aligned}$$

Since t is arbitrary, we get $E(\|.\|_\psi) \geq 4M_1^2M_2^2$.

For the opposite inequality, note that the inequality $\frac{1}{M_2} \leq \frac{\psi}{\psi_2} \leq M_1$ gives the inequality $\frac{\|.\|_2}{M_2} \leq \|.\|_\psi \leq M_1\|.\|_2$ and by Lemma 1.1 we get

$$E(\|.\|_\psi) \leq \left(\frac{M_1}{\frac{1}{M_2}}\right)^2 E(\|.\|_2) = 4M_1^2M_2^2$$

since $E(\|.\|_2) = 4$.

So we prove the equality $E(\|.\|_\psi) = 4M_1^2M_2^2$.

Let us suppose that $M_2 = \frac{\psi(\frac{1}{2})}{\psi_2(\frac{1}{2})}$. Using the same vectors as above we have

$$\|x_0 + y_0\|_\psi^2 + \|x_0 - y_0\|_\psi^2 = \frac{4[(1-t)^2 + t^2]\psi^2(\frac{1}{2})}{\psi^2(t)/2} = \frac{4\psi^2(\frac{1}{2})\psi_2^2(t)^2}{\psi_2^2(\frac{1}{2})\psi^2(t)} = \frac{4}{M_2^2\left(\frac{\psi(t)}{\psi_2(t)}\right)^2}.$$

Taking infimum over all $t \in [0,1]$ we get $f(\|.\|_\psi) \leq \frac{4}{M_1^2 M_2^2}$. For the opposite inequality, note that the inequality $\frac{1}{M_1} \leq \frac{\psi_2}{\psi} \leq M_2$ gives the inequality $\frac{\|.\|_\psi}{M_1} \leq \|.\|_2 \leq M_2\|.\|_\psi$ and by Lemma 1.1 we obtain $f(\|.\|_2) \leq M_1^2 M_2^2 f(\|.\|_\psi)$, i.e. $f(\|.\|_\psi) \geq \frac{4}{M_1^2 M_2^2}$, since $f(\|.\|_2) = 4$. So we prove the equality $f(\|.\|_\psi) = \frac{4}{M_1^2 M_2^2}$

The fact about the constant $C_{NJ}(\|.\|_\psi)$ was done in Theorem 3 from [64]. □

Remark 1.21. **1.** Very recently (2011) a paper [45] appeared dealing with $C_{NJ}(X), C_Z(X)$ and $C'_{NJ}(X) = \frac{1}{4}E(X)$. Among other results let us mention, that their Proposition 2.3 coincides in practise with the first statement of our Theorem 1.20.

2. The proof of (for instance) Theorem 1.14 can be done using similar ideas, choosing appropriate vectors x_0, y_0 in the case b). Let $t_* \in [0, \frac{1}{2}]$ such that $M_1 J(\|.\|_{\psi_*}) = f_\psi(t_*)$. In Theorem 1.14 the vectors x_0, y_0 are defined by

$$x_0 = \frac{1}{\psi(t_*)}(1 - t_*, t_*), \quad y_0 = \frac{1}{\psi(t_*)}(-t_*, 1 - t_*).$$

Next lemma (proved in [60]) was helpful in our calculation.

Lemma 1.22. *The function $\frac{\psi_r}{\psi_s}$ is non-decreasing on $[0, \frac{1}{2}]$ when $r < s \leq \infty$ and non-increasing when $s < r \leq \infty$.*

Next we consider the norm from Example 3. , namely $\max\{\|.\|_p, \lambda\|.\|_q\}$ which is shortly denoted by $\|.\|_{p,q,\lambda}$, where $1 \leq q \leq p \leq \infty$ and $\lambda \in \langle 2^{\frac{1}{p} - \frac{1}{q}}, 1\rangle$. For some particular cases of these norms, constants were calculated. For example, in [30] constants J and C_{NJ} are calculated for the norm $\|.\|_{\infty,p,\frac{1}{\lambda}}$; the constant C_{NJ} is obtained for norms $\|.\|_{2,1,\lambda}$ and $\|.\|_{2,\infty,\lambda}$ in [64]. Some general properties of the norm $\|.\|_{p,q,\lambda}$ are discussed in [59]. Here we consider constants J, C_{NJ}, C_Z, E, f and A_2 for the norm $\|.\|_{p,q,\lambda}$.

The corresponding convex function from the family Ψ_2 for the norm $||.||_{p,q,\lambda}$ is the function

$$\psi_{p,q,\lambda} = \max\{\psi_p, \lambda\psi_q\}.$$

Let $s_0 \in [0, \frac{1}{2}]$ be a point such that $\psi_p(s_0) = \lambda\psi_q(s_0)$, i.e. $[(1 - s_0)^p + s_0^p]^{1/p} = \lambda[(1 - s_0)^q + s_0^q]^{1/q}$. Then on $[0, \frac{1}{2}]$ we have

$$\psi_{p,q,\lambda}(t) = \begin{cases} \psi_p(t) & \text{for } t \in [0, s_0] \\ \lambda\psi_q(t) & \text{for } t \in [s_0, \frac{1}{2}] \end{cases}$$

which is expanded to the whole interval $[0,1]$ by symmetrization with respect to the point $\frac{1}{2}$.

a) Let $2 \le q < p \le \infty$. Since $\psi_p \le \psi_q$ and $\lambda\psi_q \le \psi_q$ we have $\psi_{p,q,\lambda} \le \psi_q$. The function

$$\frac{\psi_q}{\psi_{p,q,\lambda}}(t) = \begin{cases} \frac{\psi_q(t)}{\psi_p(t)} & \text{for } t \in [0,s_0] \cup [1-s_0,1] \\ \frac{1}{\lambda} & \text{for } t \in [s_0, 1-s_0] \end{cases}$$

attains its maximum at $\frac{1}{2}$ because $\frac{\psi_q}{\psi_p}$ is non-decreasing on $[0,1/2]$ and the maximum of $\frac{\psi_q}{\psi_{p,q,\lambda}}$ is equal to $\frac{1}{\lambda}$. So we can use Corollary 1.16 for calculating all constants.

b) Let $1 \le q < p \le 2$. From the definition it is obvious that $\psi_{p,q,\lambda} \ge \psi_p$, and by Lemma 1.22 the function $\frac{\psi_{p,q,\lambda}}{\psi_p}(t) = \begin{cases} 1 & \text{for } t \in [0,s_0] \cup [1-s_0,1] \\ \frac{\lambda\psi_q(t)}{\psi_p(t)} & \text{for } t \in [s_0, 1-s_0] \end{cases}$ is non-decreasing on $[0,1/2]$. So, it attains its maximum at $1/2$ and we can use Corollary 1.15 for calculating considered constants.

As a consequence of the above discussion we have the following result.

Theorem 1.23. *If $1 \le q < p \le 2$, then*

$$J(\|.\|_{\psi_{p,q,\lambda}}) = A_2(\|.\|_{\psi_{p,q,\lambda}}) = \lambda 2^{\frac{1}{q}}, \quad E(\|.\|_{\psi_{p,q,\lambda}}) = \lambda^2 2^{\frac{2}{q}+1},$$

$$C_{NJ}(\|.\|_{\psi_{p,q,\lambda}}) = C_Z(\|.\|_{\psi_{p,q,\lambda}}) = \lambda^2 2^{\frac{2}{q}-1}, \quad f(\|.\|_{\psi_{p,q,\lambda}}) = \lambda^{-2} 2^{3-\frac{2}{q}}.$$

If $2 \le q < p \le \infty$, then

$$J(\|.\|_{\psi_{p,q,\lambda}}) = A_2(\|.\|_{\psi_{p,q,\lambda}}) = \frac{2^{1-\frac{1}{q}}}{\lambda}, \quad E(\|.\|_{\psi_{p,q,\lambda}}) = \frac{2^{3-\frac{2}{q}}}{\lambda^2},$$

$$C_{NJ}(\|.\|_{\psi_{p,q,\lambda}}) = C_Z(\|.\|_{\psi_{p,q,\lambda}}) = \frac{2^{1-\frac{2}{q}}}{\lambda^2}, \quad f(\|.\|_{\psi_{p,q,\lambda}}) = \lambda^2 2^{1+\frac{2}{q}}.$$

Note, that our result for $E(X)$ in the case of $q=1, p=2$ (which is also prowed in [20]) completes Example 2.7 from [71] with result about $J_{X,2}(1)$.

In the case $1 \le q \le 2 \le p < \infty$ we cannot get exact value of James constant, but we can get estimates from above for it.

Theorem 1.24. *Let $q \le 2 \le p \le \infty$.*

If the distance between $\frac{1}{p}$ and $\frac{1}{2}$, is bigger than the distance between $\frac{1}{2}$ and $\frac{1}{q}$ then

$$J(\|.\|_{\psi_{p,q,\lambda}}) \le \begin{cases} \min\{\lambda 2^{1-\frac{2}{p}+\frac{1}{q}}, \frac{2^{\frac{1}{2}}\psi_2(s_0)}{\psi_p(s_0)}\} & \text{if } \lambda \in [2^{\frac{1}{p}-\frac{1}{q}}, 2^{\frac{1}{p}-\frac{1}{2}}] \\ \min\{\frac{2^{1/q}}{\lambda}, \frac{2^{\frac{1}{2}}\psi_2(s_0)}{\psi_p(s_0)}\} & \text{if } \lambda \in [2^{\frac{1}{p}-\frac{1}{2}}, 2^{\frac{1}{2}-\frac{1}{q}}] \\ \min\{\frac{2^{1/q}}{\lambda}, \frac{2^{\frac{1}{q}}\psi_2(s_0)}{\psi_q(s_0)}\} & \text{if } \lambda \in [2^{\frac{1}{2}-\frac{1}{q}}, 1]. \end{cases}$$

If the distance between $\frac{1}{p}$ and $\frac{1}{2}$, is smaller than the distance between $\frac{1}{2}$ and $\frac{1}{q}$ then

$$J(\|.\|_{\psi_{p,q,\lambda}}) \leq \begin{cases} \min\{\lambda 2^{1-\frac{2}{p}+\frac{1}{q}}, \frac{2^{\frac{1}{2}}\psi_2(s_0)}{\psi_p(s_0)}\} & if\ \lambda \in [2^{\frac{1}{p}-\frac{1}{q}}, 2^{\frac{1}{2}-\frac{1}{q}}] \\ \min\{\lambda 2^{1-\frac{2}{p}+\frac{1}{q}}, \frac{2^{\frac{1}{q}}\psi_2(s_0)}{\psi_q(s_0)}\} & if\ \lambda \in [2^{\frac{1}{2}-\frac{1}{q}}, 2^{\frac{1}{p}-\frac{1}{2}}] \\ \min\{\frac{2^{1/q}}{\lambda}, \frac{2^{\frac{1}{q}}\psi_2(s_0)}{\psi_q(s_0)}\} & if\ \lambda \in [2^{\frac{1}{p}-\frac{1}{2}}, 1]. \end{cases}$$

Consider what happens with the estimate when $\lambda \to 1$ or $\lambda \to 2^{\frac{1}{p}-\frac{1}{q}}$.

If $\lambda \to 1$ then $s_0 \to 0$ and we get that the estimate of $J(X_{\psi_{p,q,\lambda}})$ tends to $2^{1/q} = J(X_{\psi_q})$ (note that ψ_q can be regarded as a limit for $\psi_{p,q,\lambda}$ when $\lambda \to 1$).

If $\lambda \to 2^{\frac{1}{p}-\frac{1}{q}}$ then $s_0 \to \frac{1}{2}$ and we get that the estimate of $J(X_{\psi_{p,q,\lambda}})$ tends to $2^{\frac{1}{p'}} = J(X_{\psi_p})$ (note that ψ_p can be regarded as a limit for $\psi_{p,q,\lambda}$ when $\lambda \to 2^{\frac{1}{p}-\frac{1}{q}}$).

Consider constants C_{NJ} and E in the case $1 \leq q < 2 < p, \lambda \in \langle 2^{\frac{1}{2}-\frac{1}{q}}, 1\rangle \subset \langle 2^{\frac{1}{p}-\frac{1}{q}}, 1\rangle$, then for $t \in [0, 1/2]$

$$\frac{\psi_{p,q,\lambda}(t)}{\psi_2(t)} = \begin{cases} \frac{\psi_p(t)}{\psi_2(t)} & \text{for } 0 \leq t \leq s_0; \\ \frac{\lambda\psi_q(t)}{\psi_2(t)} & \text{for } s_0 \leq t \leq 1/2. \end{cases}$$

From Lemma 1.22 we get that $\frac{\psi_p}{\psi_2}$ is non-increasing and $\max_{t\in[0,s_0]} \frac{\psi_p(t)}{\psi_2(t)} = \frac{\psi_p(0)}{\psi_2(0)} = 1$ and $\frac{\lambda\psi_q}{\psi_2}$ is non-decreasing and $\max_{t\in[s_0,1/2]} \frac{\lambda\psi_q(t)}{\psi_2(t)} = \frac{\lambda\psi_q(1/2)}{\psi_2(1/2)} = \lambda 2^{\frac{1}{q}-\frac{1}{2}}$.

So $M_1 = \max_{t\in[0,1/2]} \frac{\psi_{p,q,\lambda}(t)}{\psi_2(t)} = \max\{1, \lambda 2^{\frac{1}{q}-\frac{1}{2}}\} = \lambda 2^{\frac{1}{q}-\frac{1}{2}}$ and is attained at $1/2$.

On the other hand,

$$M_2 = \max_{t\in[0,1/2]} \frac{\psi_2(t)}{\psi_{p,q,\lambda}(t)} = \max\left\{\max_{t\in[0,s_0]} \frac{\psi_2(t)}{\psi_p(t)}, \max_{t\in[s_0,1/2]} \frac{\psi_2(t)}{\lambda\psi_{q(t)}}\right\} = \frac{\psi_2(s_0)}{\lambda\psi_{q(s_0)}}.$$

By Theorem 1.20 we have

$$C_{NJ}(X_{\psi_{p,q,\lambda}}) = M_1^2 M_2^2 = 2^{\frac{2}{q}-1}\frac{\psi_2^2(s_0)}{\psi_q^2(s_0)}, \quad E(X_{\psi_{p,q,\lambda}}) = 2^{\frac{2}{q}+1}\frac{\psi_2^2(s_0)}{\psi_q^2(s_0)}.$$

Open problems: There are some problems which remains unsolved: (i) all constants for the case $1 \leq q < 2 < p,\ \lambda \in \langle 2^{\frac{1}{p}-\frac{1}{2}}, 2^{\frac{1}{2}-\frac{1}{q}}\rangle$;

(ii) constants C_Z, J, A_2 and f for the case $1 \leq q < 2 < p,\ \lambda \in \langle 2^{\frac{1}{2}-\frac{1}{q}}, 1\rangle$.

Let $1 \leq p, q \leq \infty,\ 1 \leq r < \infty,\ \alpha > 0$. Consieder the function from Example 4

$$\psi_{\alpha,p,q,r}(t) = (1+\alpha)^{-\frac{1}{r}}\Big(\psi_q^r(t) + \alpha\psi_p^r(t)\Big)^{\frac{1}{r}}$$

and the corresponding absolute normalized norm is

$$\|.\|_{\alpha,p,q,r} = (1+\alpha)^{-\frac{1}{r}}(\|.\|_q^r + \alpha\|.\|_p^r)^{\frac{1}{r}}.$$

The calculations of the constants J, C_{NJ}, E, A_2, C_Z and f for the norm $\|.\|_{\alpha,p,q,r}$ one can find in [55]. The results (after using corollaries 1.15 and 1.16) look like:

Without losing of generality we can suppose $q < p$. For the case $1 \leq q < p \leq 2$ we have

$$J(\|.\|_{\psi_{\alpha,p,q,r}}) = A_2(\|.\|_{\psi_{\alpha,p,q,r}}) = 2\psi_{\alpha,p,q,r}(\frac{1}{2}) = (1+\alpha)^{-\frac{1}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{\frac{1}{r}}.$$

$$C_{NJ}(\|.\|_{\psi_{\alpha,p,q,r}}) = C_Z(\|.\|_{\psi_{\alpha,p,q,r}}) = 2\psi^2_{\alpha,p,q,r}(\frac{1}{2}) = 2(1+\alpha)^{-\frac{2}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{\frac{2}{r}}.$$

$$E(\|.\|_{\psi_{\alpha,p,q,r}}) = 8\psi^2_{\alpha,p,q,r}(\frac{1}{2}) = 8(1+\alpha)^{-\frac{2}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{\frac{2}{r}}.$$

$$f(\|.\|_{\psi_{\alpha,p,q,r}}) = 8\psi^{-2}_{\alpha,p,q,r}(\frac{1}{2}) = 8(1+\alpha)^{\frac{2}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{-\frac{2}{r}}.$$

Let $2 \leq q < p \leq \infty$. Now we use corollary 1.16 , since $\psi_{\alpha,p,q,r} \leq \psi_q$ and $\min \frac{\psi_{\alpha,p,q,r}(t)}{\psi_q(t)}$ is attained at $t = 1/2$.

Then

$$J(\|.\|_{\psi_{\alpha,p,q,r}}) = A_2(\|.\|_{\psi_{\alpha,p,q,r}}) = \frac{1}{\psi_{\alpha,p,q,r}(\frac{1}{2})} = 2(1+\alpha)^{\frac{1}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{-\frac{1}{r}}.$$

$$C_{NJ}(\|.\|_{\psi_{\alpha,p,q,r}}) = C_Z(\|.\|_{\psi_{\alpha,p,q,r}}) = \frac{1}{2\psi^2_{\alpha,p,q,r}(\frac{1}{2})} = 2(1+\alpha)^{\frac{2}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{-\frac{2}{r}}.$$

$$E(\|.\|_{\psi_{\alpha,p,q,r}}) = \frac{2}{\psi^2_{\alpha,p,q,r}(\frac{1}{2})} = 8(1+\alpha)^{\frac{2}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{-\frac{2}{r}}.$$

$$f(\|.\|_{\psi_{\alpha,p,q,r}}) = 8\psi^2_{\alpha,p,q,r}(\frac{1}{2}) = 2(1+\alpha)^{-\frac{2}{r}}(2^{\frac{r}{q}} + \alpha 2^{\frac{r}{p}})^{\frac{2}{r}}.$$

Remark 1.25. In particular, if $p = \infty, q = 2, r = 2$, we get results of Example 7 from [30] for the $J(\|.\|_{\psi_{\alpha,p,q,r}})$ and $C_{NJ}(\|.\|_{\psi_{\alpha,p,q,r}})$ and results from [20] for $E(\|.\|_{\psi_{\alpha,p,q,r}})$ and $f(\|.\|_{\psi_{\alpha,p,q,r}})$. See also Example 2.8 from [71] in the case of $A_2(X)$ and $E(X)$.

When p, q are numbers from $[1, \infty)$, but $r = 2$, then the James and the von Neumann-Jordan constants are calculated in a paper by C.S.Yang and H.Li in 2010, using another methods.

In the very recent paper [45] - 2011, the case $r = 2$ was consiedered and results for C_{NJ}, C_Z and $C'_{NJ} = \frac{1}{4}E$ are obtained.

For the case $q \leq 2 \leq p \leq \infty$ we can get estimates in the same way as we did for the space $(\mathbf{C}^2, \|.\|_{\psi_{p,q,\lambda}})$ as we will show in the next section. But the exact values for considered constants remain undiscovered.

In the following we consider constants in two groups of concrete Banach spaces:

a) X^p spaces.

Let us remember the definition of X^p spaces. It is given usually for any $p \in \mathbf{R}$ and Banach lattice X on a measurable spaces Ω. The norm of the space X^p is given by

$$\|x\| = \||x|^p\|_X^{1/p}.$$

It is known that X^p space is a Banach lattice for $p \in (1, \infty)$.

Let X be a two-dimensional Banach spaces with absolute normalized norm $\|.\|_X$ and with a function $\psi_X \in \Psi_2$, corresponding to this norm. It is clear that the norm of X^p is absolute and normalized. Then

$$\psi_{X^p}(t) = \|(1-t,t)\|_{X^p} = \|((1-t)^p, t^p)\|_X^{\frac{1}{p}} = [(1-t)^p + t^p]^{\frac{1}{p}} \psi_X^{\frac{1}{p}} \left(\frac{t^p}{(1-t)^p + t^p} \right).$$

Since $\psi_X \leq 1$ we have $\psi_{X^p} \leq \psi_p$ and we have the following result.

Corollary 1.26. *Let $(X, \|.\|)$ is a two-dimensional Banach space with absolute normalized norm and let the corresponding function ψ_X attains its minimum at the point $t = \frac{1}{2}$. If $2 \leq p < \infty$, then*

$$J(X^p) = A_2(X^p) = \frac{1}{\psi_{X^p}(\frac{1}{2})}, \quad E(X^p) = \frac{2}{\psi^2_{X^p}(\frac{1}{2})},$$
$$C_{NJ}(X^p) = C_Z(X^p) = \frac{1}{2\psi^2_{X^p}(\frac{1}{2})} \quad f(X^p) = 8\psi^2_{X^p}(\frac{1}{2}),$$

where the function ψ_{X^p} is the convex function from Ψ_2 corresponding to the norm of the space X^p.

Proof. Since the function ψ_X attains its minimum at $\frac{1}{2}$, then $\psi_X(u) \geq \psi_X(\frac{1}{2})$ for all $u \in [0,1]$. For arbitrary $t \in [0,1]$ a number $u = \dfrac{t^p}{(1-t)^p + t^p}$ belongs to $[0,1]$ and

$$\psi_X \left(\frac{t^p}{(1-t)^p + t^p} \right) \geq \psi_X(\frac{1}{2}) = \psi_X \left(\frac{(1/2)^p}{(1-1/2)^p + (1/2)^p} \right).$$

So the function $\psi_X \left(\dfrac{t^p}{(1-t)^p + t^p} \right)$ attains its minimum at $1/2$. Then the function $\psi_X^{-\frac{1}{p}} \left(\dfrac{t^p}{(1-t)^p + t^p} \right)$ attains its maximum at $1/2$.

Since $\dfrac{\psi_p(t)}{\psi_{X^p}(t)} = \psi_X^{-\frac{1}{p}} \left(\dfrac{t^p}{(1-t)^p + t^p} \right)$ and $\psi_{X^p} \leq \psi_p$, the assumptions of Corollary 1.16 are fulfield and the results of this corollary follows as consequences of Corollary 1.16. $\square$

b) The Cesàro norm and related norms As we write in Example 5, the corresponding convex function for $p=2$ is

$$\psi_{ces}(t)=\left(\frac{4(1-t)^2}{5}+\left(\frac{1-t}{\sqrt{5}}+t\right)^2\right)^{1/2}.$$

We check that $\psi_{ces}\geq\psi_2$ and the function $\frac{\psi_{ces}}{\psi_2}$ attains its maximum at $\frac{1}{2}$. Using Corollary 1.15 we get

$$J(|.|)=A_2(|.|)=\sqrt{2+\frac{2\sqrt{5}}{5}},\quad E(|.|)=4(1+\frac{\sqrt{5}}{5}),$$

$$C_{NJ}(|.|)=C_Z(|.|)=1+\frac{\sqrt{5}}{5},\quad f(|.|)=5-\sqrt{5},$$

where $|.|$ is the corresponding norm.

Our result for $E(x)$ completes the result for $J_{X,2}(1)$ about the Cesàro space.

If parameter p is not equal to 2, then $\psi_{ces}\geq\psi_2$ but a maximum of the function $\frac{\psi_{ces}}{\psi_2}$ does not attain in $\frac{1}{2}$ and that case remains unsolved.

Consider the space with the absolute, normalized and symmetric norm

$$\|(x,y)\|_{\omega,*,ces_p}=\left(\frac{x^{*p}}{1+\omega}+\frac{\omega(x^*+y^*)^p}{1+\omega}\right)^{\frac{1}{p}}=\frac{1}{(1+\omega)^{\frac{1}{p}}}\left(\|(x,y)\|_\infty^p+\omega\|(x,y)\|_1^p\right)^{\frac{1}{p}}.$$

from Example 5 ($\omega>0$, $1\leq p<\infty$). The corresponding convex function is

$$\psi_{\omega,*,ces_p}(t)=\begin{cases}\left(\frac{(1-t)^p}{1+\omega}+\frac{\omega}{1+\omega}\right)^{1/p}, & \text{if } 0\leq t\leq\frac{1}{2};\\ \left(\frac{t^p}{1+\omega}+\frac{\omega}{1+\omega}\right)^{1/p}, & \text{if } \frac{1}{2}\leq t\leq 1.\end{cases}$$

To find the James constant (using Theorem 1.13) we calculate

$$\max_{0\leq t\leq 1/2}f_\psi(t)\max_{0\leq t\leq 1/2}\left[\frac{1-\omega^2 2^p}{(1-t)^p+\omega}+\omega 2^p\right]^{\frac{1}{p}}.$$

We get that $\max_{0\leq t\leq 1/2}f_\psi(t)=f_\psi(\frac{1}{2})$ if $\omega\leq 2^{-\frac{p}{2}}$ and $\max_{0\leq t\leq 1/2}f_\psi(t)=f_\psi(0)$ if $\omega\geq 2^{-\frac{p}{2}}$, hence

$$J(\|.\|_{\psi_{\omega,*,ces_p}})=\begin{cases}\left(\frac{1+\omega}{2^{-p}+\omega}\right)^{\frac{1}{p}} & \text{if } 0<\omega\leq 2^{-\frac{p}{2}}\\ \left(\frac{1+\omega 2^p}{1+\omega}\right)^{\frac{1}{p}} & \text{if } \omega\geq 2^{-\frac{p}{2}}\end{cases}.$$

Clearly, $J(\|.\|_{\psi_{\omega,*,ces_p}})<2$ i.e. the space with the norm $\|(x,y)\|_{\omega,*,ces_p}$ has fixed point property,Journal of Functional Analysis 233 (2006) 494 514

Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings, is a paper in J.Funct.anal. from 2006, by Jesus Garcia-Falseta, Enrique Llorens-Fuster, Eva M. Mazcunan-Navarro.

c) Consider the function $\psi(t) = (1 - t + t^a)^{\frac{1}{a}}, 1 \leq a \leq 2$. We check that it belongs to the classs Ψ_2.

The corresponding norm is $\|(x, y)\| = ((|x| + |y|)^{a-1}|x| + |y|^a)^{\frac{1}{a}}$.

If $a = 1$ we get the ℓ_1 norm, if $a = 2$ we get $\|(x, y)\| = (x^2 + |x||y| + y^2)^{\frac{1}{2}}$. Moreover, in case $a = 2$ the corresponding function $\psi(t) = \sqrt{1 - t + t^2}$ attains its minimum at the point $\frac{1}{2}$. So this can be an example for the application of Corollary 1.26. Then the corresponding space $X^p, (p \geq 2)$ has the norm $(|x|^{2p} + |x|^p|y|^p + |y|^{2p})^{\frac{1}{2p}}$. Since

$$\psi_{X^p}(t) = \|(1 - t, t)\|_{X^p} = \|((1 - t)^p, t^p)\|_X^{\frac{1}{p}} = [(1 - t)^p + t^p]^{\frac{1}{p}} \psi_X^{\frac{1}{p}} \left(\frac{t^p}{(1 - t)^p + t^p} \right).$$

$\psi_{X^p}(\frac{1}{2}) = \frac{3^{\frac{1}{2p}}}{2}$. Then by Corollary 1.26 we have that $J(X^p) = A_2(X^p) = \frac{1}{\psi_{X^p}(\frac{1}{2})} = \frac{2}{3^{\frac{1}{2p}}}$, $E(X^p) = \frac{2}{\psi^2_{X^p}(\frac{1}{2})} = \frac{8}{3^{\frac{1}{p}}}$, $C_{NJ}(X^p) = C_Z(X^p) = \frac{1}{2\psi^2_{X^p}(\frac{1}{2})} = \frac{2}{3^{\frac{1}{p}}}, f(X^p) = 8\psi^2_{X^p}(\frac{1}{2}) = 2 \cdot 3^{\frac{1}{p}}$.

d) Consider a circle with center at $(\frac{1}{2}, a)$, $a \geq \frac{3}{2}$, which passes through the points (0, 1) and (1, 1), namely $(x - \frac{1}{2})^2 + (y - a)^2 = (a - 1)^2 + \frac{1}{4}$. We will denote by $\psi_{a,c}$ the part of it, lying below the line $y = 1$, namely

$$\psi_{a,c}(t) = a - \sqrt{(a - 1)^2 + t - t^2}.$$

The corresponding norm is given by the formula

$$\|(x, y)\|_{a,c} = a(|x| + |y|) - \sqrt{(a - 1)^2(x^2 + y^2) + [2(a - 1)^2 + 1]|xy|}.$$

We check that $\psi_{a,c} \in \Psi_2$, $\psi_{a,c} \geq \psi_2$ and the max $\frac{\psi_{a,c}(t)}{\psi_2(t)}$ is attained at $\frac{1}{2}$. Applying Corollary 1.15 with $\psi = \psi_{a,c}, \psi_p = \psi_2$ we get

$$J(\|.\|_{a,c}) = A_2(\|.\|_{a,c}) = 2\psi_{a,c}\left(\frac{1}{2}\right) = 2\left(a - \sqrt{a^2 - 2a + \frac{5}{4}}\right)$$

and the corresponding formulas for $E(\|.\|_{a,c})$, $f(\|.\|_{a,c})$ and $C_{NJ}(\|.\|_{a,c}) = C_Z(\|.\|_{a,c})$.

2. Some Properties of Interpolation Spaces Constructed for More than Two Spaces

2.1. **Preliminaries about interpolation spaces.** First interpolation theorem in operator theory for L_p spaces was obtained by M.Riesz in 1926 as an inequality for bilinear forms. It was specified and its operator variant was given by G.O. Thorin in 1939, Lund, Sweden. For integral operators in Orlicz spaces an interpolation theorem was established in 1934 by W.Orlicz (in an underestimated paper). Next serious contribution was J.Marcinkiewicz interpolation theorem (1939), which proof was published in 1956 by A.Zygmund . In the fifties of twentieth century I.Stein and G.Weiss obtained different generalizations of Riesz-Thorin and Marcinkiewicz theorems, but all generalizations concerned L_p spaces, Orlicz spaces and spaces close to them. The development of general interpolation theorems for abstract Hilbert and Banach spaces began independently in 1958 in different contries. The first publications belong to Lions, Gagliardo, Calderon, S.G.Krein, Aronszajn (in the period 1958-1961.) Later on, significant rhole played the works of J.Peetre.

In the next years interpolation theory was developed intensively by many groups of authors and found important applications in theory of function spaces, partial defferential equations, Fourier series, approximation theory, etc..

Let us mention, that the most important interpolation methods can be regarded as apparat for generalization of the classical theorems roufgly speaking, for Riesz-Thorin theorem - the complex method of interpolation, and for Marcinkiewicz theorem - real methods.

In what follows $\bar{A}$ will be a couple (A_0, A_1) of Banach spaces , a finite family $(A_1, A_2, \dots, A_n)$ or infinite family $A(\gamma), \gamma \in \Gamma$.

When we write $T \in L(\overline{A}, \overline{B})$ we mean that the linear map T acts from $\Sigma\overline{A}$ to $\Sigma\overline{B}$ and its restrictions $T/A(\gamma)$ is a bounded operator from $A(\gamma)$ to $B(\gamma)$. The goal of interpolation theory is to give methods of construction of interpolation spaces A and B so that if $T \in L(\overline{A}, \overline{B})$, then T is bounded from the space A to the space B.

2.1.1. *Case of Banach couples.* In the theory of interpolation (see [6]) one usually considers Banach couples, i.e. pairs $\bar{A} = (A_0, A_1)$ such that A_0 and A_1 are Banach spaces embedded in a common topological vector space U.

Usually the intersection $\Delta\overline{A} = A_0 \cap A_1$ is provided with the norm $\|a\|_{\Delta\overline{A}} = \max(\|a\|_{A_0}, \|a\|_{A_1})$, the sum $\Sigma\overline{A} = A_0 + A_1$ consists of those elements of U, which can be represented as $a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1$ and its norm is $\|a\|_{A_0+A_1} = \|a\|_{\Sigma\overline{A}} = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + \|a_1\|_{A_1})$. Let $\bar{A}$ and $\bar{B}$ be two Banach couples. Let $\Delta\bar{A} \subset A \subset \Sigma\bar{A}, \;\; \Delta\bar{B} \subset B \subset \Sigma\bar{B}$. The spaces A and B are called to be interpolation for couples $\bar{A}$ and $\bar{B}$ if from $T \in L(\bar{A}, \bar{B})$ (sometimes written $T : \bar{A} \to \bar{B}$)it follows that $T : A \to B$ is bounded.

If we have the inequality

$$\|T\|_{A\to B} \leq \max(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1})$$

we say that A and B are exact interpolation spaces. If the inequality is only $\|T\|_{A\to B} \leq C\max(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1})$ we say that A nd B are uniform interpolation spaces. Note that $\Delta\bar{A}$ and $\Delta\bar{B}$ are exact interpolation spaces, the same for $\Sigma\bar{A}$

and $\Sigma\bar{B}$. We say that A and B are interpolation spaces of type θ if

$$\|T\|_{A\to B} \leq C(\|T\|_{A_0\to B_0})^{1-\theta}(\|T\|_{A_1\to B_1})^{\theta},$$

if C=1 , i.e. we have log-convexity inequality, then we say that A and B are exact interpolation spaces of type θ.

One of the most important among the various constructions of interpolation with respect to a given couple is the complex method leading to the spaces $[A_0, A_1]_\theta$ (where $0 < \theta < 1$) and the real method leading to the spaces $(A_0, A_1)_{\theta,q}$ (where $0 < \theta < 1$ and $0 < q \leq \infty$). This spaces are exact interpolation spaces of type θ.

Consider now the interpolation space, constructed by the complex interpolation method. As usualy, denote $\Pi = \{z : 0 < Rez < 1\}$ and $\overline{\Pi} = \{z : 0 \leq Rez \leq 1\}$. Let $\overline{A} = (A_0, A_1)$ be a Banach couple of complex space. $F(A_0, A_1)$ will be the set of functions $f : \overline{\Pi} \to A_0 + A_1$, with properties

1) $f(z)$ is continuous and bounded in the norm of $A_0 + A_1$;

2) $f(z)$ is analytic in the norm $A_0 + A_1$ in Π ;

3) $f(it)$ is continuous and bounded in the norm of A_0 ,

$f(1 + it)$ is continuous and bounded in the norm of A_1.

Denote

$$\|f\|_{F(A_0,A_1)} = \max\{\sup \|f(it)\|_{A_0}, \sup \|f(1 + it)\|_{A_1}\}.$$

Let $0 \leq \theta \leq 1$. The space of complex interpolation method is defined like

$$[A_0, A_1]_\theta = \{a \in A_0 + A_1; a = f(\theta) \text{ for some } f \in F(A_0, A_1)\}$$

with norm

$$\|a\|_{[A_0,A_1]_\theta} = \inf_{f(\theta)=a, f\in F(A_0,A_1)} \|f\|_{F(A_0,A_1)}$$

(shortly $\|a\|_\theta$.)

The space, constructed by the complex interpolation method gives an example of exact interpolation spaces of type θ.

Now we present some real interpolation methods. Though the first appeared the real interpolation spaces called spaces of means and spaces of traces (this theory was developed by Lions and Peetre in 1964), but much more popular the real K- and J- methods became.

The Peetre K-functional is defined like

$$K(t, a) = K(t, a, \overline{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),\ a \in \Sigma\bar{A} = A_0 + A_1.$$

The interpolation space, constructed by K-method

$$A_{\theta,q,K} = \{a \in \Sigma\overline{A} : \|a\|_{\theta,q,K} = \Phi_{\theta,q}(K(t,a)) = \left(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{dt}{t}\right)^{1/q} < \infty\}$$

for admisssible values of θ and q.

When define the space of J-method one works not with $\Sigma\overline{A}$, but with $\Delta\overline{A} = A_0 \cap A_1$. For $t > 0$ and $a \in \Delta\overline{A}$ the J-functional is defined to be

$$J(t,a) = J(t,a,\overline{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1}).$$

The space $A_{\theta,q,J}$ consisits of elements of $\Sigma\overline{A}$, which can be presented like

$$a = \int_0^\infty u(t)\frac{dt}{t}$$

(covergence in $\Sigma\overline{A}$), here $u(t)$ is measurable in $\Sigma\overline{A}$ function, with values in $\Delta\overline{A}$, for which $\Phi_{\theta,q}(J(t,u(t))) = \left(\int_0^\infty (t^{-\theta}J(t,u(t)))^q \frac{dt}{t}\right)^{1/q} < \infty$. For the norm we have

$$\|a\|_{A_{\theta,q,J}} = \inf \Phi_{\theta,q}(J(t,u(t))),$$

and inf is taken over all functions, satisfying the presentation of a from above, and also the inequality. Usually $0 < \theta < 1,\ 1 \leq q \leq \infty$ and $0 \leq \theta \leq 1, q = 1$.

If $0 < \theta < 1,\ 1 \leq q \leq \infty$, then $A_{\theta,q,J} = A_{\theta,q,K}$ with equivalent norms, in this situation often the notation $A_{\theta,q}$ is used, if $\theta = 0,1$ and $q = \infty$ we agree $A_{\theta,q} = A_{\theta,q,K}$.

2.1.2. *Case of finite ($n > 2$) and infinite family of Banach spaces.* Parts of the theory concerning interpolation between t

wo Banach spaces can be generalized to cover also cases where one interpolates between finitely many Banach spaces and even between general families of (infinite many) Banach spaces. Yet in some of the first works of J.-L.Lions in 1961 a generalization of the complex interpolation method was suggested in the case of n banach spaces $A_1, A_2, \ldots, A_n$ embedded in a Hausdorff linear topological spaces. For instance, in the case of three spaces A_1, A_2, A_3 the domain $\Omega = \{(z,w) : Rez \geq 0, Rew \geq 0, 0 \leq Rez + Rew \leq 1\}$ and functions which are continuous and bounded in Ω , analytic in the interior of Ω in the norm of $A_1 + A_2 + A_3$, such that $f(it, i\tau) \in A_0, f(1+it, i\tau) \in A_2, f(it, 1+i\tau) \in A_3$. Then a space $F(A_1, A_2, A_3)$ is defined and interpolation space $[A_1, A_2, A_3]_{\theta,\rho}$ for $(\theta,\rho) \in \Omega, [\theta,\rho] \in R^2$. More detiailed research of the suggested interpolation method was done by A.Fawini in 1971, using the apparatus of analytic functions of more than one variable . S.G.Krein from Voronezh, Russia, my supervisor for my Ph.D.Thesis, suggested me a problem: to make a construction for interpolation in the case of n Banach spaces avoiding use of analytic functions of more than one variable, because sometimes the calculations and theory looked heavy and even clumsy. He advised me to consider analytic functions in $\{z : |z| < 1\}$ and $f(e^{it}) \in A_1$ for $e^{it} \in \gamma_1, \ldots, f(e^{it}) \in A_n$ for $e^{it} \in \gamma_n$, where $\bigcup_{j=1}^{n} \gamma_j = \{z : |z| = 1\}$. this resulted in the paper [46] in 1978. This paper was a basis of the next work with S.G.Krein, (as a joke, he called this "running from Favini to infinity") where instead of n Banach spaces we considered an infinite family af Banach spaces, rougly speaking we consider the space A_t "to sit at the point e^{it}. So in 1980, 1982 our papers [34] [35] appeared where interpolation space A_z was constructed. At the same time (1979, 1980, 1982) in St.Louis a big group of five mathematicions (including G.Weiss)see [19] suggested similar construction for family of Banach spaces, which became more popular, than ours. Jaak Peetre used for the spaces, suggested by the group of 5 mathematicians the abreviation "St.Louis space" and for our with S.G.Krein - "Voronez spaces" see for instance [61]. A new nontrivial application of this theory was found by Ferenczi as a further development of some important works of Maurey and Gowers.

In 1993 Maurey and Gowers constructed the first known example of a space without an unconditional basic sequence. Their space, has even the stronger property of being hereditarily indecomposable. A Banach space X is said to be hereditarily indecomposable (or H.I.) if no infinite-dimensional subspace of X is decomposable, that is, no infinite-dimensional subspace of X can be written as a topological direct sum of two infinite-dimensional subspaces.

In other words, a space X is H.I. if for any infinite-dimensional subspaces Y and Z of X, any $\varepsilon > 0$, there exist vectors $y \in Y, z \in Z$, such that $\|y\| = \|z\| = 1$ and $\|y - z\| \leq \varepsilon$. Later on, W. T. Gowers showed the following dichotomy theorem: every Banach space X contains a subspace that is either spanned by an unconditional basis, or is hereditarily indecomposable. Because of this theorem, it is of particular interest to know about general properties of H.I. spaces. The space constructed by Gowers and Manrey is reflexive, however it is not uniformly convex. In his article Ferenczi [24] provides an example of a complex uniformly convex hereditarily indecomposable space, using a Gowers-Maurey type construction and the theory of complex interpolation for a family of Banach spaces of Coifman, Cwikel, Rochberg, Sagher, and Weiss mentioned above.

Let us mention also a paper by J.Pisier from 2010 [63], in which the idea of complex interpolation in families of Banach spaces is used to get some results from geometry of Banach spaces , for instance to relax the notion of $\theta-$ Hilbertian spaces.

We will only mention here that in 1971 Yoshikawa published a paper where a generalization of the methods of means and traces of Lions and Peetre was suggested for the case of more than two spaces. A theory of real interpolation between n-tuples of Banach spaces, based on the K and J functionals, was worked out by Sparr in 1974 . A parallel theory of interpolation between 2^k-tuples of Banach spaces was developed by Fernández in 1979. Lately Cobos and Peetre (1991) have developed a theory which, in particular, covers both of the constructions of Sparr and Fernández with $n = 3$ and $n = 4$ respectively. On the other hand, even earlier, the construction of Sparr had been extended by Cwikel and Janson [21] to the case of interpolation between a fairly general family $\bar{A} = (A_t)_{t \in \Gamma}$, where the A_t are Banach spaces $A_t \subset U, t \in \Gamma$ and $Z(t)$ is a probability measure(such a family is called bounded family). Moreover, the spaces $L_M(A, Z), U_M(A, Z), \Lambda_M(A, Z)$ are defined there, when M is one of the following interpolation methods : FL- the complex method of Favini-Lions for finite families, the St.L - the complex method for infinite families of St.Louis group, J_p and K_p - the real methods, defined by Sparr. Some new real interpolations methods for families of Banach spaces were introduced and studied by M.J.Carro in [10], [11](continuous method - 1994, 1996, connected to continuous variants of K and J functionals) and by me and L.-E.Persson in [51], [48], and [52] (discrete methods - 1991, 1993 and 1996-connected to the discrete variants of K and J functionals). Having in mind what happened with the complex interpolation for families, where "Voronezh" spaces went on the second plane and became not popular, here, J.Peetre, (he worked at that time with M.J.Carro on preparing of [16]) suggested to join our efforts. L.-E.Persson invited M.J.Carro and me to Sweden and in 1997 the paper [14] by M.-J.Carro, J.Peetre, L.-E.Persson and me appeared. There these methods were compared and some new (limit construction) methods were introduced. The collaboration continued and some papers, with coathors M.J.Carro and me , and in some also L.-E.Persson appeared - see [12], [13] where we work with continuous K- and J-functionals, in [15]- with discrete K- and J- functionals.

Let us mention also some cases in which real interpolation of finite and infinite families can have nice applications. Begin with the three-space approach , interpolation in triples of Banach spaces can be interesting for instance in the case of smooth functions. In [3] it was shown, for the case of Sparr interpolation spaces, that in contrast to the case of couples of Banach spaces (n=2) for which even the scale of Besov spaces is not stable under interpolation, for triples stability in the frame of Besov spaces, based on Lorentz spaces appears. Moreover, results about interpolation of triples of weghted Lebesque spaces and a variant of reiteration theorem is used to extend the Stein-Weiss interpolation theorem, known for $L_p(\mu)$-spaces with change of measures to Lorentz spaces with a change of measure. In particular this shows that for some preoblems in analysis the three-space real approach is really more useful than the usual real interpolation between couples.

As an important consequence of the results from our paper [13], which deals with families of Banach spaces we can give a new proof of the following result for couples: $T : A_{\theta,p} \to B_{\theta,p}$ is compact whenever $T : (A_0, A_1) \to (B_0, B_1)$ is an interpolation operator so that $T : A_0 \to B_0$ is compact. This result has remained an open question for several years till M. Cwikel gave in 1992 a positive answer. With our tecniques we can give the insight of this particular problem.

When we speak about interpolation in families of Banach spaces (complex or real) we are in the situation when the actual family of Banach spaces is indexed by the points of the unit circle $\Gamma = \{|z| = 1\}$ in the complex plane, while the interpolation spaces are labeled by the points of the unit disk $D = \{|z| < 1\}$.

We will consider an i.f. - interpolation family (in the sense of [19]) $\bar{A} = \{A(\gamma),\ \gamma \in \Gamma\}$, where $\Gamma = \{|z| = 1\}$ is the unit circle, such that if $U = U_{\bar{A}}$ is the containing space, then $\|a\|_U \le \|a\|_{A(\gamma)}$ for every $a \in A(\gamma)$ and for a. e. $\gamma \in \Gamma$.

We suppose that for each $a \in \bigcap_{\gamma\in\Gamma} A(\gamma)$ the function $\|a\|_\gamma$ is measurable. As in [19] the log-intersection of the family is defined like $\mathtt{A} = \{a \in A(\gamma)$ for a.e. $\gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma \, dg < \infty\}$. The sum space $\Sigma_\gamma A_\gamma$ is defined in [36] as a set of all elements $a \in U$ that can be written as $a = \Sigma_\gamma a_\gamma$ with $a_\gamma \in A_\gamma$ and $\Sigma_\gamma \|a_\gamma\|_{A_\gamma} < \infty$. $\Sigma_\gamma A_\gamma$ is a Banach space with the norm $\|a\|_\Sigma = \inf \Big\{ \sum_\gamma \|a_\gamma\|_{A_\gamma} \Big\}$, where the infimum is taken over all representations of the element $a = \sum_\gamma a_\gamma$, the series converges in U and $a_\gamma \in A(\gamma)$ (note that only countable many summands a_γ are different from zero).

When speaking about interpolation of (infinite) families of Banach spaces we have to mention the complex methods (see [19]), [34] and [35]), they appear before the real methods. In the terms of [19], let denote the complex interpolation space $A(z)$, $|z| < 1$, constructed for the family $\bar{A} = \{A(\gamma),\ \gamma \in \Gamma\}$. The interpolation theorems say that if the linear map T acts from $\Sigma\bar{A}$ to $\Sigma\bar{B}$, its restrictions $T/A(\gamma)$ is bounded operator from $A(\gamma)$ to $B(\gamma)$ and T maps $\mathtt{A}$ to $\bigcap B_\gamma$ and $\|Ta\|_{B_\gamma} \le M(\gamma)\|a\|_{A_\gamma}$ then T maps the interpolation space $A(z)$ into $B(z)$ and the norm there is less or equal to

$$M(z) = \exp\Big(\int_\Gamma \log\ M(\gamma)\, P_z(\gamma) d\gamma\Big),$$

P_z being the Poisson kernel. This can be regarded as an infinite variant of the inequality (log convexity inequality) in the notion of the exact interpolation method of type θ (here z) in the case when the families just consist of just two spaces (case of Banach couples). As we will see the situation of the real interpolation for infinite families is more complicated and the norm of the operator, acting between the interpolation spaces is estimated by the so called Dicesar function, which in some situations can be calculated and in special cases it is equal to $M(z)$. More pleasant is the situation when the family is $n-$ tuple, i.e. consists of n Banach spaces.

For each $a \in U$, I consiedred in 1988, in [49] the discrete K-functional

$$K^{(1)}(\alpha, a) = \inf \left\{ \sum_j \alpha(\gamma_j) \|a_{\gamma_j}\|_{A_{\gamma_j}} \right\},$$

where the infimum is taken over all representations of the element $a = \sum_j a_{\gamma_j}$ with convergence in U and $a_{\gamma_j} \in A(\gamma_j)$. For each $a \in \mathtt{A}$, we also define the J-functional by $J^{(1)}(\alpha, a) = \sup_\gamma \alpha(\gamma)\|a\|_\gamma \cdot$.

Another K-functional was defined by M.J.-Carro in 1994 (see [10], [11]) like

$$K^{(2)}(\alpha, a) = \inf \left\{ \left(\int_\Gamma \alpha(\gamma)\|a(\gamma)\|_\gamma \, d\gamma \right\},$$

and the infimum extends over all representations $a = \int_\Gamma a(\gamma)\, d\gamma$ (convergence in U) with $a(\cdot) \in \bar{G}$.

Recall from [10] that

$$G = \left\{ b = \sum_{\text{finite}} b_j \chi E_j : b_j \in \mathtt{A} \text{ and } E_j \text{ pairwise disjoint measurable sets in } \Gamma \right\},$$

where $\mathtt{A}$ is the log-intersection of the family and χ_E denotes the characteristic function of E, while $a(\cdot) \in \bar{G}$ means that $a(\cdot)$ is a Bochner integrable function in U such that $a(\gamma) \in A_(\gamma)$ a.e. $\gamma \in \Gamma$ and such that $a(\cdot)$ can be approximated a.e. in the $A(\cdot)$-norm by functions $a_n(\cdot)$ belonging to G.

Similarly, $J^{(2)}(\alpha, a) = \text{esssup}_\gamma \alpha(\gamma)\|a\|_\gamma$.

We will give definition of interpolation spaces based on this two variants of K- and J-functionals, informaly we call them "discrete" in $j = 1$ and "continuous" in $j = 2$ cases. Later on when in case $j = 2$ appears "esssup" shall be interpreted as "sup" in the discrete case $j = 1$. Moreover, "essinf" and "a.e." shall be interpreted as "inf" and "everywhere" when working with the "discrete" method.

Let S be a set of measurable functions $\alpha : \Gamma \to R^+$ .We will suppose that it is a multiplicative group of bounded (in discrete case) and essentialy bounded (in continuous case) functions.

Remember, $\alpha(z) = \exp\big(\int_\Gamma \log\ \alpha(\gamma)\, P_z(\gamma) d\gamma\big)$, P_z being the Poisson kernel.

Now we are going to consider real interpolation spaces with variants of K- and J-functionals , in which an index q, $1 \leq q \leq \infty$ appears. For shortness we will consider only the "continuous" case, it is clear what will appear in the "discrete" case.

For $1 \leq p, q \leq \infty$, the following interpolation spaces were defined in [11]:

$$(A)^S_{z_0,p,q;K} = \left\{ a \in U;\ \left(\sum_{\alpha \in S} \left(\frac{K_q(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} < +\infty \right\},$$

where

$$K_q(\alpha, a) = \inf \left\{ \left(\int_\Gamma \big(\alpha(\gamma) \|a(\gamma)\|_\gamma\big)^q\, d\gamma \right)^{1/q} \right\},$$

and the infimum extends over all representations $a = \int_\Gamma a(\gamma)\, d\gamma$ (convergence in U) with $a(\cdot) \in \bar{G}$.

Moreover, J-interpolation space is defined as

$$(A)^S_{z_0,p,q;J} = \left\{ a = \sum_\alpha a_\alpha;\ a_\alpha \in \mathtt{A};\ \left(\sum_{\alpha \in S} \left(\frac{J_q(\alpha, a_\alpha)}{\alpha(z_0)} \right)^p \right)^{1/p} < +\infty \right\},$$

where

$$J_q(\alpha, a) = \left(\int_\Gamma \big(\alpha(\gamma) \|a\|_\gamma\big)^q\, d\gamma \right)^{1/q}.$$

Furthermore the corresponding norm is given by

$$\|a\|_{(A)^S_{z_0,p,q;J}} = \inf \left(\sum_{\alpha \in S} \left(\frac{J_q(\alpha, a_\alpha)}{\alpha(z_0)} \right)^p \right)^{1/p},$$

where the infimum extends over all representations $a = \sum_\alpha a_\alpha$ (convergence in U) with $a_\alpha \in \mathtt{A}$.

In fact, in order to be precize, here we would write $(A)^{S,2}_{z_0,p,q;K}$ instead of $(A)^S_{z_0,p,q;K}$ and $(A)^{S,2}_{z_0,p,q;J}$ instead of $(A)^S_{z_0,p,q;J}$, because we use "continuous" variants of K- and J-functionals.

If we use for $K_q(\alpha, a)$ not $K^2_q(\alpha, a)$ as we did above, but
$K^1_q(\alpha, a) = \inf \left\{ \sum_j \alpha(\gamma_j) \|a_{\gamma_j}\|_{A_{\gamma_j}} \right\}$, we get the definition of the space $(A)^{S,1}_{z_0,p,q;K}$. Analoguously we give the definition of $(A)^{S,1}_{z_0,p,q;J}$.

Athough no restrictions are needed on S for the definition of these spaces, some restrictions are needed to ensure good properties of these spaces. It can be seen in [10] and [11] that the natural conditions on S to ensure that $(A)^S_{z_0,p,q;K}$ and $(A)^S_{z_0,p,q;J}$ are interpolation Banach spaces such that we have the usual embedding

$(A)^S_{z_0,p,q;J} \subset (A)^S_{z_0,p,q;K}$ are S to be a multiplicative group so that $\alpha^{-1} \in L^\infty$ and and so called size condition: 1) If for every $a \in \mathtt{A}$, $J(1,a) < \infty$ for every $z_0 \in D$

$$\sum_{\alpha \in S} \frac{\text{essinf}_{\gamma \in \Gamma} \alpha(\gamma)}{\alpha(z_0)} < \infty.$$

2) If $J(1,a) = \infty$ for some $a \in \mathtt{A}$, then we need a stronger condition, that there exists a compact set $K \subset D$ such that

$$\sum_{\alpha \in S} \frac{\text{essinf}_{z \in K} \alpha(z)}{\alpha(z_0)} < \infty.$$

The size condition has its natural and popular analogue in the methods for finite families.

Also, observe that these interpolation spaces remain the same if we change the family in a set of zero measure and since they are the same if we replace $A(\gamma)$ by the closure of $\mathtt{A}$ in $A(\gamma)$ we can assume without lost of generality that $\mathtt{A}$ is dense in all $A(\gamma)$.

Let us note, that in Carro's paper [10] instead of $K_q(\alpha, a)$ it stays $K(\alpha, a)$, which is $K_q(\alpha, a)$ with q=1 , and $J(\alpha, a) = J_\infty(\alpha, a)$ instead of $J_q(\alpha, a)$. In [10] the spaces $(A)^S_{z_0,1,q;K}$ are $(A)^S_{z_0,\infty,q;J}$ are considered. The cases for general q appear in the next paper in a less popular journal. We observe, that although in the classical situations of Sparr, Fernandez and Cobos-Peetre K- method spaces do not depend on q ([11], [12]), in general they are not equivalent for different q's. And for instance the weakly compactness and compactness properties are preserved by interpolation for every $q > 1$ but not in the case $q = 1$. The same for the J-method and $q = \infty$. Analogues situation appears also when one considers some geometrical properties, like uniform convexity, for instance, as we will see later.

As we mentioned, in [10] the interpolation spaces $(A)^S_{z_0,p,q;K}$ were considered, where $q = 1$. In [14] we denote them as $(A)^{S,2}_{z_0,p;K}$, the notation $(A)^{S,1}_{z_0,p;K}$ is given for the variant of the definition where instead of continuous K-functional is used the discrete one. Analoguous definitions could be given for the J-method. In [14] the so called K- and J- classes of interpolation spaces are consiedered.

Under some natural condition we proved there that the spaces $(A)^S_{z_0,p,q,K}$ and$(A)^S_{z_0,p,q,J}$ belong to the class $K^S_{z_0,q}(\overline{A})$. The complex interpolation space $A[z_0]$ is also from this class. We will not point our attention to this part of the paper, because we will mention analoguous things for the very general case of Cwikel-Janson $K(A,Z)$ and $J(A,Z)$.

Here we will present only two theorems from [14], concerning comparison of the continuous and discrete variants of K-interpolation spaces.

If some of the indexes are not important at the moment, we will omit them, for instance writing $(A)^{S,1}_{K}$ instead of $(A)^{S,1}_{z_0,p;K}$.

Theorem 2.1. *For any bounded interpolation family $\overline{A}$ one has $(A)^{S,2}_{K} \subset (A)^{S,1}_{K}$.*

Proof. It is sufficient to establish that $K^{(1)}(\alpha, a) \le K^{(2)}(\alpha, a)$ for every $\alpha \in S$ and every $a \in (A)^{S,2}_{K}$. It follows from the definition of $K^{(2)}(\alpha, a)$ that for any $\varepsilon > 0$ we can find a function $a(.) \in \overline{G}$ such that $a = \int_\Gamma a(\gamma)\, d\gamma$ and $a = \int_\Gamma \|a(\gamma)\|_{A_\gamma}\, d\gamma \le (1+\varepsilon)K^{(2)}(\alpha, a)$. We also choose a sequence $a_n(.) \to a(.)$ and write $a_n = \int_\Gamma a_n(\gamma)\, d\gamma \in \mathtt{A}$. Let us note for a moment $\Sigma = \Sigma_\gamma \alpha(\gamma) A(\gamma)$. For m and n big enough we have

$$\|a_n - a_m\|_\Sigma \le \int_\Gamma \|a_n(\gamma) - a_m(\gamma)\|_\Sigma\, d\gamma \le \int_\Gamma \|a_n(\gamma) - a_m(\gamma)\|_{A_\gamma} \alpha(\gamma)\, d\gamma \le \varepsilon.$$

Hence $\{a_n\}$ converges to an element b in Σ and since this space is embedded in U we have $b = a$. We conclude that $a \in \Sigma$ and moreover that taking n sufficently large for any $\varepsilon > 0$

$$\|a\|_\Sigma = K^{(1)}(\alpha, a) \le \|a_n\|_\Sigma + \varepsilon \le \int_\Gamma \alpha(\gamma)\|a_n(\gamma)\|_{A_\gamma}\, d\gamma + \varepsilon$$

$$\le \int_\Gamma \alpha(\gamma)\|a(\gamma)\|_{A_\gamma}\, d\gamma + 2\varepsilon \le (1+\varepsilon)K^{(2)}(\alpha, a) + 2\varepsilon.$$

Let $\varepsilon \to 0$ and the proof is complete. □

Next we note that the methods $(A)^{S,1}_{K}$ and $(A)^{S,2}_{K}$ do not coincide in general. For example, if S consists of the one function $\alpha = 1$, then we have $(A)^{S,1}_{K} = \Sigma_\gamma A(\gamma)$ and if $B(\gamma) = A(\gamma)$ a.e. on Γ, while $B(\gamma) \ne A(\gamma)$ on a set of measure, then in general $(B)^{S,1}_{K} = \Sigma_\gamma B(\gamma) \ne (A)^{S,1}_{K}$, but $(B)^{S,2}_{K} = (A)^{S,2}_{K}$. However, the situation changes if we restrict ourselves to countable families.

Theorem 2.2. *If $\overline{A}$ is countable (i.e. if $A(\gamma) = A_n$ for all $\gamma \in \Gamma_n$, where $\{\Gamma_n\}_{n\in N}$ is a partition into intervals of Γ such that $\mathtt{A}$ is dense in $A(\gamma)$ for every $\gamma \in \Gamma$ and all functions $\alpha \in S$ are regular (i.e. every point is a Lebesque point of α), then $(A)^{S,1}_{K} = (A)^{S,2}_{K}$.*

Proof. According to the previous theorem it is obviously sufficient to prove that $K^{(2)}(\alpha, a) \le K^{(1)}(\alpha, a)$ for every $\alpha \in S$ and every $a \in (A)^{S,1}_{K}$. By definition of $K^{(1)}$ we can find for every $\varepsilon > 0$ elements $a_n \in A_n$ such that $a = \sum_n a_n$ and $\sum_n \inf_{\Gamma_n} \|a_n\|_{A_n} \le (1+\varepsilon)K^{(1)}(\alpha, a)$.

Denote by C_n the subset of Γ_n on which $\alpha(\gamma) \le 1+\varepsilon)\mathrm{essinf}_{\Gamma_n}\alpha(\gamma)$. It is clear, that $\mu(C_n) > 0$. Construct a function $\phi_n \ge 0$, with a support in C_n, such that

$\int_{\Gamma_n} \phi_n(\gamma)\, d\gamma = 1$. Then

$$\int_{\Gamma_n} \alpha(\gamma)\phi_n(\gamma)\, d\gamma = \int_{C_n} \alpha(\gamma)\phi_n(\gamma)\, d\gamma$$

$$\leq (1+\varepsilon)\text{essinf}\alpha(\gamma)_{\Gamma_n} \int_{C_n} \phi_n(\gamma)\, d\gamma = (1+\varepsilon)\text{essinf}_{\Gamma_n}\alpha(\gamma).$$

Suppose $\text{essinf}_{\Gamma_n}\alpha(\gamma) > \inf_{\Gamma_n}\alpha(\gamma)$, hence there exists $\gamma_0,\ \alpha(\gamma_0) < \text{essinf}\alpha(\gamma)$ and there exists a number $B_1 > \alpha(\gamma_0)$ such that if $A_{B_1} = \{\gamma : \alpha(\gamma) \leq B_1\}$, then $\mu(A_{B_1}) = 0$. Since every point is a Lebesque point, we have

$$\alpha(\gamma_0) = \lim_{r\to 0} \frac{1}{\mu(B(\gamma_0,r))} \int_{B(\gamma_0,r)} \alpha(\gamma)\, d\gamma = \lim_{r\to 0} \frac{1}{\mu(B(\gamma_0,r))} \int_{B(\gamma_0,r)-A_B} \alpha(\gamma)\, d\gamma$$

$$\geq \frac{B_1\mu(B(\gamma_0,r)-A_B)}{\mu(B(\gamma_0,r))} = B_1$$

We came to a contrudiction with the conjecture, that $\text{essinf}_{\Gamma_n}\alpha(\gamma) > \inf_{\Gamma_n}\alpha(\gamma)$. Let us take

$$a(\gamma) = \sum_n a_n\phi_n(\gamma)\chi_{\Gamma_n}(\gamma)\text{with} \int_{\Gamma_n} \phi_n(\gamma)\, d\gamma = 1$$

and

$$\int_{\Gamma_n} \alpha(\gamma)\phi_n(\gamma)\, d\gamma \leq (1+\varepsilon)\text{essinf}_{\Gamma_n}\alpha(\gamma) = (1+\varepsilon)\inf_{\Gamma_n}\alpha(\gamma).$$

So we find that

$$K^{(2)}(\alpha, a) \leq \int_{\Gamma} \alpha(\gamma)\|a(\gamma)\|_{A_\gamma}\, d\gamma = \sum_n \|a_n\|_{A_n} \int_{\Gamma_n} \alpha(\gamma)\phi_n(\gamma)\, d\gamma$$

$$\leq (1+\varepsilon)\sum_n \|a_n\|_{A_n} \inf_{\Gamma_n}\alpha(\gamma) \leq (1+\varepsilon)^2 K^{(1)}(\alpha, a).$$

Let $\varepsilon \to 0$ and the proof is complete. □

Remark 2.3. Each of the interpolation spaces $(A)_K^{S,1}$ and $(A)_K^{S,2}$ constitutes a generalization of the constructions of Cobos-Peetre, Fernandez and Sparr.

Let us give several examples of the above mentioned interpolation spaces. We suppose that the intersection space ΔA_j is dense in every A_j.

(I) Let $\overline{A} = (A_0, A_1)$; that is, $A(\gamma) = A_j$ for $\gamma \in \Gamma_j$, $j = 0, 1$ with $\{\Gamma_0,\ \Gamma_1\}$ a partition of Γ. Then if we take $S = \{\alpha_n(\gamma), n \in Z\}$, where $\alpha_n(\gamma) = \left\{ \begin{array}{ll} 1, & \gamma \in \Gamma_0 \\ 2^n, & \gamma \in \Gamma_1 \end{array} \right\}$,

we get that, $(A)_{z_0,p;K}^S = (A)_{z_0,p;J}^S \equiv (A_0, A_1)_{|\Gamma_1|_{z_0},p}$ since $K^{(1)}(\alpha_n, a) \sim K(2^n, a)$ and $J_\infty(\alpha_n, a) \sim J(2^n, a)$ with $K(2^n, a)$ and $J(2^n, a)$ the classical K and J functionals.

(II) If $\overline{A} = (A_0, A_1, \cdots, A_m)$; that is, $A(\gamma) = A_j$ for $\gamma \in \Gamma_j$, $j = 0, \cdots, m$, $\{\Gamma_0, \cdots, \Gamma_m\}$ a partition of Γ and we consider $S = \{\alpha_{\overline{n}}(\gamma)\}$ where

$$\alpha_{\overline{n}}(\gamma) = \left\{ \begin{array}{cc} 1, & \gamma \in \Gamma_0 \\ 2^n, & \gamma \in \Gamma_j \end{array} \right\}, \text{ with } j = 1, \cdots, m, \ \overline{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$$

then

$$(A)^S_{z_0,p;K} \equiv (A_0, A_1, \cdots, A_N)_{(|\Gamma_j|_{z_0}, j=1,\cdots,N),p;K}$$

(Sparr K-space, see [**10**]). Analogously for the J-method.

(III) If $\overline{A} = (A_0, A_1, A_2, A_3)$; that is, $A(\gamma) = A_j$ when $\gamma \in \Gamma_j$, $j = 0, \cdots, 3$ and we consider $S = \{\alpha_{\overline{n}}(\gamma)\}$ where $\alpha_{\overline{n}}(\gamma) = \left\{ \begin{array}{cc} 1, & \gamma \in \Gamma_0 \\ 2^n, & \gamma \in \Gamma_1 \\ 2^k, & \gamma \in \Gamma_2 \\ 2^n 2^k, & \gamma \in \Gamma_3 \end{array} \right\}$, with $\overline{n} = (n, k) \in \mathbb{Z}^2$ then $(A)^S_{z_0,p;K} \equiv (A_0, A_1, A_2, A_3)_{(\theta_1,\theta_2),p;K}$ where $\theta_1 = |\Gamma_1 \cup \Gamma_3|_{z_0}$ and $\theta_2 = |\Gamma_2 \cup \Gamma_3|_{z_0}$ (Fernández K-space, see [**7**]). Analogously for the J-method.
(IV) If $\overline{A} = (A_1, \cdots, A_m)$ and we take

$$S = \left\{ \alpha_{n,k}(\gamma) = \left\{ (a_j)^n (b_j)^k : \gamma \in \Gamma_j, \ j = 1, \cdots, m \right\}, \ (n, k) \in \mathbb{Z}^2 \right\}$$

with $a_j = 2^{x_j}$ and $b_j = 2^{y_j}$ and (x_j, y_j) the vertices of a polygon in the affine plane $\mathbb{R}^2$, then $(A)^S_{z_0,p;K} = (A_1, \cdots, A_m)_{(\alpha,\beta),p;K}$ where $(\alpha, \beta) = \sum_{j=1}^m |\Gamma_j|_{z_0}(x_j, y_j)$ (Cobos-Peetre interpolation spaces, see [**6**]). Analogously for the J-method.

Here we will give the original definition of Cobos-Peetre method for finite families, namely method of polygons.

Let $\Pi = \overline{P_1 P_2 \ldots P_n}$ be a convex polygon with vertices $P_i(x_i, y_i), i = 1, \ldots, n$. $t, s > 0$ Cobos and Peetre define the K- and J- functionals like

$$K(t, s, a, \overline{A}) = \inf \left\{ \sum_{i=1}^n t^{x_i} s^{y_i} \|a_i\|_{A_i} : \ a = \sum_{i=1}^n a_i, a_i \in A_i \right\}$$

$$J(t, s, a, \overline{A}) = \max_{1 \le i \le n} \{ t^{x_i} s^{y_i} \|a\|_{A_i} \}.$$

Let (α, β) be a point from the interior of Π. When $1 \le q \le \infty$ the interpolation space $A_{(\alpha,\beta),q,K}$ is defined like the set of those $a \in \Sigma\overline{A}$, for which the norm

$$\|a\|_{A_{(\alpha,\beta),q,K}} = \left\{ \int_0^\infty \int_0^\infty [t^{-\alpha} s^{-\beta} K(t, s, a, \overline{A})]^q \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{q}}$$

is finite. The space $A_{(\alpha,\beta),q,J}$ is defined like the set of those $a \in \Sigma\overline{A}$, which can be represented in the form $a = \int_0^\infty \int_0^\infty u(t, s) \frac{dt}{t} \frac{ds}{s}$, with $u(t, s)$ strongly measurable and for which $\left\{ \int_0^\infty \int_0^\infty [t^{-\alpha} s^{-\beta} J(t, s, u(t, s), \overline{A})]^q \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{q}} < \infty$. The norm $\|a\|_{A_{(\alpha,\beta),q,J}}$ is given like inf of the above value over all representation of a. In the case of 3 spaces, when Π is the simplex with vertices $(0,0),(0,1),(1,0)$, and $\vec{\theta}=(1-\alpha-\beta,\alpha,\beta)$ $\alpha>0,\beta>0,\alpha+\beta<1$ this are actually the Sparr spaces, when $n=4$ - Fernandez spaces.

In 1990 [48] I suggested a variant to generalize the polygon method to the case of infinitely many spaces, i.e. in families $A_\gamma,\gamma\in[0,2\pi)$. I define the K-functional like

$$K(t,s,a,\overline{A_\gamma})=\inf\left\{\sum_\gamma t^{\cos\gamma}s^{\sin\gamma}\|a_\gamma\|_{A_\gamma}:\ a=\sum a_\gamma,\ a_\gamma\in A_\gamma\right\}.$$

For $|\theta|<1$ the K-method space $A_{\theta,q,K}$ is defined like the set of all elements $a\in\sum A_\gamma$, having finite norm

$$\|a\|_{A_{\theta,q,K}}=\left\{\int_0^\infty\int_0^\infty[t^{-Re\theta}s^{-Im\theta}K(t,s,a,\overline{A_\gamma})]^q\frac{dt}{t}\frac{ds}{s}\right\}^{\frac{1}{q}}.$$

The space $A_{\theta,q,J}$ can be defined in a proper way. More about these spaces, and also for the classes $K_\theta(A_\gamma)$ and $J_\theta(A_\gamma)$ and interpolation results - in [52].

Let us mention the Sparr definition of interpolation spaces for n-tuples, here we will write in details for K- method. Later on we will consider some concrete cases of Sparr's spaces for triples.

Let $A_0,A_1,...,A_n$ be $n+1$ Banach or quasi-Banach spaces. We will say that they form a compatible collection or simply a collection $\vec{A}=(A_0,A_1,...,A_n)$ if they are linearly and continuously embedded in some (common for all) topological linear space with Hausdorff topology. Then we can, analogously to the case of couple, define the $K-$functional (see [66]) by the formula:

$$K(\vec{t},a;\vec{A})=\inf(\|a_0\|_{A_0}+t_1\|a_1\|_{A_1}+...+t_n\|a_n\|_{A_n}),\ \vec{t}=(t_1,...,t_n)\in R_n^+,\quad(2.1)$$

where inf is taken over all decompositions $a=a_0+a_1+...+a_n$.

Let $\vec{\theta}=(\theta_0,\theta_1,...,\theta_n)$ be a parameter vector, i.e. $\theta_i>0$ and $\theta_0+\theta_1+...+\theta_n=1$ and let $0<q\le\infty$. Then the interpolation space $\vec{A}_{\vec{\theta},q}=(A_0,A_1,...,A_n)_{\vec{\theta},q}$ (usually this space is defined by $\vec{A}_{\vec{\theta},q;K}$ is defined by the norm (or quasinorm, for simplicity we always say norm)

$$\|a\|_{\vec{\theta},q}=(\int_{\mathbb{R}_+^n}(t_1^{-\theta_1}t_2^{-\theta_2}...t_n^{-\theta_n}K(\vec{t},a;\vec{A}))^q\frac{dt_1}{t_1}\frac{dt_2}{t_2}...\frac{dt_n}{t_n})^{\frac{1}{q}}\quad(2.2)$$

with the usual modification for $q=\infty$. As the $K-$functional is a concave function on $\mathbb{R}_+^n$, therefore the norm (2.2) can be written in an equivalent form

$$\|a\|_{\vec{\theta},q}\approx(\sum_{(i_1,i_2,...,i_n)\in\mathbb{Z}^n}(2^{-\theta_1i_1}2^{-\theta_2i_2}...2^{-\theta_ni_n}K(2^{i_1},2^{i_2},...,2^{i_n},a;\vec{A}))^q)^{\frac{1}{q}}.$$

We will call a parameter vector $\vec{\theta}$ an extended parameter vector if some of its coordinates θ_i could be equal to zero. In this case in the definition of the $K-$functional and norms we omit the spaces A_i, parameters t_i and integrate on the set of a smaller dimension. In a particular case, when $\vec{\theta}$ has one, let say $i-$th, coordinate equal to one, and therefore all other coordinates are equal to zero, by $(A_0, A_1, ..., A_n)_{\vec{\theta},q}$ we will mean the space A_i.

2.1.3. *The Dicesar function.* The Dicesar function plays a fundamental role in our papers [13] and [15]. We mentioned it in the Introduction, but here we will consider the case when S is not a group and now we give a very general definition of Dicesar function:

Let S be a subset of $\widetilde{L} = \{\alpha : \Gamma \to R^+, \alpha$ measurable and $\log \alpha \in L^1(\Gamma)\}$, let $z_0 \in D$ and let $M \in \widetilde{L}$. The Dicesar function is then defined by

$$D_{z_0}^S(M) = \inf_{\alpha \in \widetilde{S}} \left\{ \frac{\sup \alpha(\gamma) M(\gamma)}{\alpha(z_0)} \right\},$$

where $\widetilde{S}$, is the set of all $\beta \in \widetilde{L}$ so that, for every $\alpha \in S$, $\alpha\beta^{-1} \in S$ and the mapping $\alpha \to \alpha\beta^{-1}$ is injective.

Obviously if S is a multiplicative group $\widetilde{S} = S$ and this is just the Dicesar function we have talked about in the Introduction. Also, $\widetilde{S}$ is always non-empty since $1 \in \widetilde{S}$, for every S.

Let us recall (from the introduction) that we can get an estimate of the operator norm $\|T\|_{F(\bar{A}) \to F(\bar{B})}$ (where F is either $(\cdot)_{z_0;K}^S$ or $(\cdot)_{z_0;J}^S$ and S is a multiplicative group) by the Dicezar function.

The Dicezar function is a generalization of the function $M_0^{1-\theta} M_1^\theta$ for the classical case of two Banach spaces. Other special cases of the Dicesar function are and $D_{(\alpha,\beta)}(M_1, \cdots, M_n)$, suggested by Cobos and Peetre in [18] in the finite family case and its generalization $D_\theta(M)$ for infinite family , suggested by me and L.E.Persson in [52].

For $M_1, M_2, \ldots, M_n \geq 0$ in the case of mentioned above polygon method Cobos and Peetre set

$$D_{(\alpha,\beta)}(M_1, M_2, \ldots, M_n) = \inf_{t,s>0} \{ \max_{1 \leq i \leq n} [t^{x_i - \alpha} s^{y_i - \beta} M_i] \}.$$

We introduce

$$E_{(\alpha,\beta)}(M_1, M_2, \ldots, M_n) = \inf_{t,s>0} \left\{ \sum_1^n t^{x_i - \alpha} s^{y_i - \beta} M_i \right\}.$$

Clearly

$D_{(\alpha,\beta)}(M_1, M_2, \ldots, M_n) \leq E_{(\alpha,\beta)}(M_1, M_2, \ldots, M_n) \leq n D_{(\alpha,\beta)}(M_1, M_2, \ldots, M_n)$, I obtained (in [48] using minimization of the function $f(t,s) = M_1 t^{\alpha-1} s^\beta +$

$M_2 t^\alpha s^{\beta-1} + M_3 t^\alpha s^\beta$ over the simplex Π with vertices $(0,0),(0,1),(1,0)$, and $\vec{\theta} = (1-\alpha-\beta,\alpha,\beta)$ $\alpha > 0, \beta > 0, \alpha+\beta < 1)$ that $E_{(\alpha,\beta)}(M_1,M_2,M_3) = \frac{M_1^\alpha M_2^\beta M_3^{1-\alpha-\beta}}{\alpha^\alpha \beta^\beta (1-\alpha-\beta)^{1-\alpha-\beta}}$. The functions $D_{(\alpha,\beta)}(M_1,M_2,\ldots,M_n)$ and $E_{(\alpha,\beta)}(M_1,M_2,\ldots,M_n)$ appear in Theorems 1.and 2. in my paper [48] where the measures of non-compactness of operators acting in $n-$tuples of Banach spaces are estimated.

Since $1 \in S$, $D^S_{z_0}(M) \leq \|M\|_\infty$ and obviously $D^S_{z_0}(M) \geq M(z_0)$. Moreover, if $M \in S$, $D^S_{z_0}(M) = M(z_0)$. Finally, the connection between the function $D^S_{z_0}(M)$ and the classes $K^S_{z_0}$ and $J^S_{z_0}$ was studied in our paper [14], where also some concrete examples of the Dicesar function were computed.

The Dicesar function plays a fundamental role in our papers [13] and [15]. As an example (when S is not a group), let us take for some $\alpha \in \widetilde{L}$, $S^+ = \{\alpha^n;\ n \in \mathbb{N}\}$ and $S^- = \{\alpha^n;\ n \in \mathbb{Z}^-\}$, then $\widetilde{S^+} = S^-$. In particular, if

$$S = \left\{ \alpha_n(\gamma) = \left\{ \begin{array}{ll} 1, & \text{if } \gamma \in \Gamma_0; \\ 2^n, & \text{if } \gamma \in \Gamma_1. \end{array} \right\}, n \in Z \right\}$$

with $\{\Gamma_0,\ \Gamma_1\}$ a partition of Γ and $M(\gamma) = M_j$ if $\gamma \in \Gamma_j$, one can check that

$$D^{S^+}_{z_0}(M) = \inf_{n \leq 0} \frac{\max(M_0, 2^n M_1)}{2^{n|\Gamma_1|_{z_0}}} \sim M_0^{1-\theta} M_1^\theta \inf_{n \leq 0} \frac{\max(1, 2^{n+k})}{2^{(n+k)|\Gamma_1|_{z_0}}},$$

where $2^k \leq M_0/M_1 \leq 2^{k+1}$. Hence, if $k \leq 0$, we get that $D^{S^+}_{z_0}(M) \sim M_0$, while if $k > 0$, $D^{S^+}_{z_0}(M) \sim M_0^{1-\theta} M_1^\theta$.

As we mentioned above $\|T\|_{(\bar{A})^S_{z_0,F} \to (\bar{B})^S_{z_0,F}}$ when F is K or J space, can be estimated by the Dicesar function. Next we consider the case when operator acts from interpolation space of the J-method to interpolation space of the K-method. The Theorem proved below is proved and used in our paper with M.J.Carro [13](not included in the present work) about interpolation of compactness property, where we proved the so called one sided compactness result, when the condition about compactness of the operator is only on a subset of, but not on the whole Γ. We are going to mention compactness in the following subsection.

Theorem 2.4. *Let S_1 and S_2 be two sets in $\tilde{L}$. Then, for any i.f. (interpolation family) $\overline{A}$ and $\overline{B}$ and any interpolation operator T such that $\|T\|_{A(\gamma) \to B(\gamma)} \leq M(\gamma)$, for every $\gamma \in \Gamma$, we have that*

(a) If $p \geq 1$, then

$$\|T\|_{(A)^{S_1}_{z_0,p,;J} \to (A)^{S_2}_{z_0,p;K}} \leq \left(\sup_{\beta \in S_1} \sum_{\alpha \in S_2} \left[\left(\sum_{\lambda \in S_1} \frac{\inf_{\gamma \in \Gamma} (\alpha \lambda^{-1} M)(\gamma)}{\alpha(z_0) \lambda^{-1}(z_0)} \right)^{p-1} \frac{\inf_{\gamma \in \Gamma} (\alpha \beta^{-1} M)(\gamma)}{\alpha(z_0) \beta^{-1}(z_0)} \right] \right)^{1/p},$$

(b) If $p < 1$, then

$$\|T\|_{(\bar{A})^{S_1}_{z_0,p;J}\to(\bar{A})^{S_2}_{z_0,p;K}} \le \left(\sup_{\beta\in S_1}\sum_{\alpha\in S_2}\left(\frac{\inf_{\gamma\in\Gamma}(\alpha\beta^{-1}M)(\gamma)}{\alpha(z_0)\beta^{-1}(z_0)}\right)^p\right)^{1/p}.$$

Proof. The proof will be done for the discrete K- and J-methods, remember $K^{(1)}(\alpha,a) = \inf\left\{\sum_j \alpha(\gamma_j)\|a_{\gamma_j}\|_{A_{\gamma_j}}\right\}$, where the infimum is taken over all representations of the element $a = \sum_j a_{\gamma_j}$ with convergence in U and $a_{\gamma_j} \in A(\gamma_j)$ and $J^{(1)}(\alpha,a) = \sup_\gamma \alpha(\gamma)\|a\|_\gamma$ for each $a \in \mathtt{A}$.

We will write here just $K(\alpha,a), J(\alpha,a)$ instead of $K^{(1)}(\alpha,a), J^{(1)}(\alpha,a)$.

Let $\alpha \in (\bar{A})^{S_1}_{z_0,p;J}$. Given $\varepsilon > 0$, let $a = \sum_{\beta\in S_1} a_\beta$, such that $a_\beta \in \mathtt{A}$ and

$$\left(\sum_{\beta\in S_1}\left(\frac{J(\beta,a_\beta)}{\beta(z_0)}\right)^p\right)^{1/p} \le \|a\|(\bar{A})^{S_1}_{z_0,p;J} + \varepsilon\cdot$$

Let us write $a_\beta = a_\beta \sum_j \varphi(\gamma_j)$ with $\varphi \in L^\infty$ so that $\sum_j \varphi(\gamma_j) = 1$. Then,

$$\begin{aligned}
K(\alpha,Ta) &\le K(\alpha M,a) \le \sum_{\beta\in S_1} K(\alpha M,a_\beta)\\
&\le \sum_{\beta\in S_1}\inf\left\{\sum \alpha(\gamma_j)M(\gamma_j)\beta^{-1}(\gamma_j)\beta(\gamma_j)\|a_\beta\|_{\gamma_j}\varphi(\gamma_j) : \sum_j \varphi(\gamma_j) = 1\right\}\\
&\le \sum_{\beta\in S_1}\inf_\Gamma(\alpha M\beta^{-1})(\gamma)J(\beta,a_\beta)\cdot
\end{aligned}$$

Case $p \ge 1$.

Note that

$$\|Ta\|^p_{\bar{A}^{S_2}_{z_0,p;K}} = \sum_{\alpha\in S_2}\left(\frac{K(\alpha,Ta)}{\alpha(z_0)}\right)^p \le \sum_{\alpha\in S_2}\left(\sum_{\beta\in S_1}\frac{\inf_\Gamma(\alpha\beta^{-1}M)(\gamma)}{\beta^{-1}(z_0)\alpha(z_0)}\frac{J(\beta,a_\beta)}{\beta(z_0)}\right)^p.$$

For a while denote

$$A_\beta = \frac{J(\beta,a_\beta)}{\beta(z_0)},\quad B_{\alpha,\beta} = \sum_{\beta\in S_1}\frac{\inf_\Gamma(\alpha\beta^{-1}M)(\gamma)}{\beta^{-1}(z_0)\alpha(z_0)} \text{ and } C_\alpha = \left(\sum_{\lambda\in S_1} B_{\alpha,\lambda}\right)^{p-1}.$$

We have $\left(\sum_{\beta\in S_1} A^p_\beta\right)^{1/p} \le \|a\|_{(\bar{A})^{S_1}_{z_0,p;J}} + \varepsilon\cdot$

Then, using Holder inequality we estimate

$$\sum_{\alpha\in S_2}\left(\sum_{\beta\in S_1} A_\beta B^{\frac{1}{p}}_{\alpha,\beta}B^{\frac{1}{p'}}_{\alpha,\beta}\right) \le \sum_{\alpha\in S_2}\left[\left(\sum_{\beta\in S_1} A^p_\beta B_{\alpha,\beta}\right)\left(\sum_{\beta\in S_1} B_{\alpha,\beta}\right)^{\frac{p}{p'}}\right]$$

$$= \sum_{\alpha\in S_2}\left(\sum_{\beta\in S_1} A^p_\beta B_{\alpha,\beta}\right)C_\alpha = \sum_{\beta\in S_1}\left(\sum_{\alpha\in S_2} A^p_\beta B_{\alpha,\beta}C_\alpha\right) = \sum_{\beta\in S_1} A^p_\beta\left(\sum_{\alpha\in S_2} B_{\alpha,\beta}C_\alpha\right)$$

$$\leq \sup_{\beta\in S_1}\left(\sum_{\alpha\in S_2} B_{\alpha,\beta}C_\alpha\right)\sum_{\beta\in S_1} A_\beta^p \leq$$

$$\leq (\|a\|_{(\bar{A})^{S_1}_{z_0,p;J}}+\varepsilon)\left(\sup_{\beta\in S_1}\sum_{\alpha\in S_2}\left[\left(\sum_{\lambda\in S_1}\frac{\inf_{\gamma\in\Gamma}(\alpha\lambda^{-1}M)(\gamma)}{\alpha(z_0)\lambda^{-1}(z_0)}\right)^{p-1}\frac{\inf_{\gamma\in\Gamma}(\alpha\beta^{-1}M)(\gamma)}{\alpha(z_0)\beta^{-1}(z_0)}\right]\right)$$

From this we get the result by letting ε tend to zero.

Case $p<1$.

Here we will use the converse Minkovski inequality and estimate

$$\|Ta\|^p_{(\bar{A})^{S_2}_{z_0,p;K}} = \sum_{\alpha\in S_2}\left(\frac{K(\alpha,Ta)}{\alpha(z_0)}\right)^p \leq \sum_{\alpha\in S_2}\left(\sum_{\beta\in S_1}\frac{\inf_\Gamma(\alpha\beta^{-1}M)(\gamma)}{\beta^{-1}(z_0)\alpha(z_0)}\frac{J(\beta,a_\beta)}{\beta(z_0)}\right)^p$$

$$=\sum_{\alpha\in S_2}\left(\sum_{\beta\in S_1}A_\beta B_{\alpha,\beta}\right)^p \leq \sum_{\alpha\in S_2}\sum_{\beta\in S_1}A_\beta^p B_{\alpha,\beta}^p = \sum_{\beta\in S_1}\sum_{\alpha\in S_2}A_\beta^p B_{\alpha,\beta}^p$$

$$\leq \sup_{\beta\in S_1}\left(\sum_{\alpha\in S_2}B^p_{\alpha,\beta}\right)\sum_{\beta\in S_1}A_\beta^p \leq$$

$$\sup_{\beta\in S_1}\sum_{\alpha\in S_2}\left(\frac{\inf_\Gamma(\alpha\beta^{-1}M)(\gamma)}{\beta^{-1}(z_0)\alpha(z_0)}\right)^p(\|a\|_{(\bar{A})^{S_1}_{z_0,p;J}}+\varepsilon)^p$$

From this we get the result by letting ε tends to zero. $\square$.

Remark 2.5. The above general estimate give us a sufficient condition to have the embedding

$$(\bar{A})^{S_1}_{z_0,p;J}\subset(\bar{A})^{S_2}_{z_0,p;K}$$

when applying it to the identity operator.

In particular, if $S_1=S_2=S$ and S is a multiplicative group, the above embedding holds if S satisfies the size condition. This was proved in [10].

Some other examples can be the couples (S^-,S^+), (S^+,S^+), (S^-,S^-) and (S^+,S^-), when S as in Examples (I) and (II) from the previous subsection.

In [15] the "mixed" Dicesar function $D^S_{z_0,p;J\to K}(M)$ was defined. In practice, this is the right hand side of the estimation in theorem, in the case $S_1=S_2$.

Proposition 3.6 from [15] asserts:

$$D^S_{z_0,p;J\to K}(M)\leq D^S_{z_0}(M)D^S_{z_0,p;J\to K}(1)\cdot$$

More about the Dicesar function and related results can be found in our paper with M.J.Carro and L.-E.Persson [15] where the Dicesar function, controls the norms of some interpolation operators, in particular some non-linear operators, operators of weakened type (A,Ψ), correct operators acting in some Banach lattices, summing operators. Yet in [51] we have a result about the Dicesar function for C-subadditive operators in Cobos-Peetre spaces. The Dicesar function is also involved when we estimate the measure of noncompactness - [48].

2.2. **Measure of non compactness, compact, limited operators, weakened type operators.** Here we will comment some of the results from [50], namely those which concern limited operators and measure of non-compactness. We do not include here results about weakly compact operators from the paper [12] because of the big volume of this paper and because we will consider measure of weak non-compactness results in the third section. Let us mention that we use the notation $\Sigma A_t = \Sigma_{t\in\Gamma} A_t$ instead of the used by Cwikel and Janson [21] $\sup_{t\in\Gamma} A_t$ and $\Delta A_t = \Delta_{t\in\Gamma} A_t$ instead of the used by Cwikel and Janson [21] $\inf_{t\in\Gamma} A_t$.

Let $h(t)$ be $Z-$measurable function. After [21]

$$K(h(t), a, A_t) = \inf \sum h(t_j)\|a_{t_j}\|_{A_{t_j}}$$

where inf is taken over all representation of $a = \sum a_{t_j}, a_{t_j} \in A_{t_j}, \sum \|a_{t_j}\|_{A_{t_j}} < \infty$.

We say that a Banach space E belongs to the class $K(A, Z)$ iff $E \subset \Sigma A_t$ and for any $Z-$measurable function $h(t)$ bounded from above and below by positive constants, the inequality

$$K(h(t), a, A_t) \leq C \exp\left(\int_\Gamma \log h(t) dZ(t)\right) \|a\|_E$$

holds.

For $a \in \Delta A_t$ the generalized J-functional is defined by

$$J(h(t), a, A_t) = \sup_{t\in\Gamma} h(t)\|a\|_{A_t}.$$

We say that a Banach space F belongs to the class $J(A, Z)$ iff $\Delta A_t \subset F$ and the inequality

$$\|a\|_F \leq C \exp\left(\int_\Gamma \log h^{-1}(t) dZ(t)\right) J(h(t), a, A_t)$$

holds for all $a \in \Delta A_t$ and for any $Z-$measurable function $h(t)$ bounded from above and below by positive constants.

Using Theorem 2.2.1 [21] and a modification of Theorem 2 [49] we get in [50] that when $U = \Sigma A_t$, then $L_M(A, Z)$ and $U_M(A, Z)$ belong to both classes $K(A, Z)$ and $J(A, Z)(\Lambda_M(A, Z)$ in general may be not complete), here M is one of the following interpolation methods: FL- the complex method of Favini-Lions for finite families, the St.L - the complex method for infinite families of St.Louis group, J_p and K_p - the real methods, defined by Sparr.

Remember, that a subset E of the Banach space E is called limited(or more precisely, limited in A) if $\lim_{n\to\infty}\sup_{x\in E}|x_n^*(x)| = 0$ for every weak*-null sequence x_n^* in A^*, the dual space to A.

A bounded linear operator $T : A \to B$ is called limited if T maps the unit ball U_A (and thus every bounded subset of A) to a limited subset of B.

Theorem 2.6. *Let $A_t, t \in \Gamma$ be a bounded family of Banach spaces, B, an arbitrary Banach space and let $A \subset K(A, Z)$. Let γ be a subset of Γ with positive measure. Suppose that $T : \sum A_t \to B$, $\sup \|T/A_t\|_{A_t\to B_t} < \infty$ and T is a limited operator from $\sum_\gamma A_t$ into B. Then T is a limited operator, acting from A into B.*

Proof. Let $M(t)$ be a bounded by positive constants Z-measurable function such that $M(t) \geq \|T/A_t\|_{A_t\to B_t}$. Let $M(t) \geq m$ and an arbitrary $\varepsilon > 0$ is given. We define a function $h(t) = M(t)$ when $t \in \gamma$ and $h(t) = M(t)/\varepsilon$ when $t \in \Gamma\backslash\gamma$.

Let $x \in U_A$. According to the definition of generalized K-functional there exists a representation $x = \sum x_{t_j}, x_{t_j} \in A_{t_j}$, such that

$$\sum_{t_j\in\Gamma} h(t_j)\|x_{t_j}\|_{A_{t_j}} \leq 2K(h(t), x, A_t).$$

Having in mind that $A \subset K(A, Z)$, we get that

$$\sum_{t_j\in\Gamma} h(t_j)\|x_{t_j}\|_{A_{t_j}} \leq 2C \exp\left(\int_\Gamma \log h(t) dZ(t)\right).$$

Let y_n^* be weak*-null sequence in B^*, then there exists a constant $C_1 > 0$ such that $\sup \|y_n^*\|_{B^*} \leq C_1$. We are going to show that $T(U_A)$ is a limited set in B. We have to estimate

$$\overline{\lim_{n\to\infty}} \sup_{y\in T(U_A)} |y_n^*(y)| \leq \overline{\lim_{n\to\infty}} \sup_{x\in U_A} \left| y_n^*\left(T\sum_{t_j\in\gamma} x_{t_j}\right)\right| + \overline{\lim_{n\to\infty}} \sup_{x\in U_A} \left| y_n^*\left(T\sum_{t_j\in\Gamma\backslash\gamma} x_{t_j}\right)\right|.$$

Let $x^0 = \sum_{t_j\in\gamma} x_{t_j}, x^0 \in \sum_{t\in\gamma} A_t$, and its norm there could be estimated.

$$\|x^0\|_{\sum_{t\in\gamma} A_t} \leq \sum_{t_j\in\gamma} \|x_{t_j}\|_{A_{t_j}} \leq \frac{1}{m}\sum_{t_j\in\Gamma} h(t_j)\|x_{t_j}\|_{A_{t_j}} \leq \frac{2}{m} C \exp\left(\int_\Gamma \log h(t) dZ(t)\right) = K.$$

The image of the ball $U_{\sum_{t\in\gamma} A_t}(K)$ (of radius K) is a limited set in B and hence

$$\overline{\lim_{n\to\infty}} \sup_{x\in U_A} \left| y_n^*\left(T\sum_{t_j\in\gamma} x_{t_j}\right)\right| < \varepsilon.$$

On the other hand

$$\overline{\lim_{n\to\infty}} \sup_{x\in U_A} \left| y_n^* \left(T \sum_{t_j\in\Gamma\setminus\gamma} x_{t_j} \right) \right| \le \overline{\lim_{n\to\infty}} \sup_{x\in U_A} \sum_{t_j\in\Gamma\setminus\gamma} |y_n^*(Tx_{t_j})| \le \sum_{t_j\in\Gamma\setminus\gamma} \overline{\lim_{n\to\infty}} \sup_{x\in U_A} |y_n^*(Tx_{t_j})|$$

$$\le C_1\varepsilon \sum_{t_j\in\Gamma\setminus\gamma} \|T\|_{A_{t_j}\to B} \|x_{t_j}\|_{A_{t_j}} \frac{1}{\varepsilon} \le C_1\varepsilon \sum_{t_j\in\Gamma\setminus\gamma} \frac{M(t_j)}{\varepsilon} \|x_{t_j}\|_{A_{t_j}} \le C_1\varepsilon \sum_{t_j\in\Gamma} h(t_j)\|x_{t_j}\|_{A_{t_j}}$$

$$\le C_1\varepsilon 2K(h(t), x, A_t) \le C_1\varepsilon \left(\exp\left(\int_\Gamma \log h(t) dZ(t) \right) \right).$$

Since we have

$$\int_\Gamma \log h(t)dZ(t) = \int_\gamma \log M(t)dZ(t) + \int_{\Gamma\setminus\gamma} logM(t)dZ(t) - \int_{\Gamma\setminus\gamma} \log\varepsilon dZ(t)$$

$$\le \sup_{t\in\Gamma} \log M(t) \int_\gamma dZ(t) - \log\varepsilon \int_{\Gamma\setminus\gamma} dZ(t) = \log \sup_{t\in\Gamma} M(t) - (1-\mu_Z(\gamma))\log\varepsilon,$$

where $\mu_Z(\gamma) = \int_\gamma dZ(t)$,

Hence $C_1\varepsilon \left(\exp\left(\int_\Gamma \log h(t)dZ(t)\right)\right) \le C_2\varepsilon^{\mu_Z(\gamma)}$. Since ε is arbitrary small we obtain $\lim_{n\to\infty} \sup_{y\in T(U_A)} |y_n^*(y)| = 0$ and hence TU_A is a limited set in B.

Definition 2.7. Let A be a complex Banach space and let E be a bounded subset of A. The Kuratowski measure of noncompactness of $E, \psi_A(E)$ is defined by $\psi_A(E) = \inf\{\varepsilon > 0 : E$ can be covered by finitely many sets of diameter $\varepsilon\}$.

Let $k \ge 0$, then a map $T : A \to B$ is called a k-set of contraction iff $\psi_B(T(E)) \le k\psi_A(E)$ for all bounded sets E and $\beta(T) = \min\{k : T$ is a k-set contraction$\}$ is called the measure of noncompactness of T. Another measure of noncompactness of an operator $\tilde{\beta}(T)$ is defined analoguously to $\beta(T)$using the ball measure of non-compactness $\tilde{\psi}_A$ and $\tilde{\psi}_B$. Remember $\tilde{\psi}_A(E) = \inf\{\varepsilon > 0 : E$ can be covered by finitely many balls of radius $\varepsilon\}$. Let us note that $\frac{1}{2}\beta(T) \le \tilde{\beta}(T) \le 2\beta(T)$. The following assertion is a generalization of Theorem 1 from [68].

Theorem 2.8. *1) Let $A_t, t \in \Gamma$ be a bounded family of Banach spaces, B an arbitrary Banach space and let the Banach space A belong to the class $K(A, Z)$. Let γ_i be Z-measurable subsets of Γsuch that $\bigcup_{i=1}^n \gamma_i = \Gamma$. Suppose that $T : \sum A_t \to B$, $\sup \|T/A_t\|_{A_t\to B_t} < \infty$ .Then*

$$\beta(T_{A\to B}) \le C \prod_{i=1}^n [\mu_Z(\gamma_i)]^{-\mu_Z(\gamma_i)} \left[\beta\left(T_{\sum_{\gamma_i} A_t\to B}\right)\right]^{\mu_Z(\gamma_i)},$$

where $\mu_Z(\gamma_i) = \int_{\gamma_i} dZ(t)$.

2) Let $A_t, t \in \Gamma$ be a bounded family of Banach spaces, B an arbitrary Banach space and let the Banach space A belong to the class $J(A,Z)$. Suppose that $T : B \to A_t$, $\sup \|T\|_{B\to A_t} < \infty$ .Then

$$\beta(T_{B\to A}) \le C \prod_{i=1}^{n} \left[\beta\left(T_{B\to\Delta_{\gamma_i} A_t}\right)\right]^{\mu_Z(\gamma_i)}.$$

Proof. Denote $k_i = \beta\left(T_{\sum_{\gamma_i} A_t \to B}\right)$ and let $h(t)$ be a step function admiting values $m_i = k_i/\mu_Z(\gamma_i)$ on γ_i, Ω - a bounded subset of A and $\varepsilon > 0$. Since $A \in K(A,Z)$ there exists a representation $a = \sum_{t_j \in \Gamma} a_{t_j}, a_{t_j} \in A_{t_j}$, such that

$$\sum_{t_j\in\Gamma} h(t_j)\|a_{t_j}\|_{A_{t_j}} \le (1+\varepsilon)K(h(t),a,A_t) \le (1+\varepsilon)C\exp\left(\int_\Gamma \log h(t)dZ(t)\right)\|a\|_A = M\|a\|_A$$

where $M = (1+\varepsilon)C\prod_{i=1}^n m_i^{\mu_Z(\gamma_i)}$.

Denote by Ω_i the set of all elements $a_i$of the form $a_i = \sum_{t_j\in\gamma_i} a_{t_j}, a_{t_j} \in A_{t_j}$.

Obviously $\Omega_i \subset \sum_{\gamma_i} A_t$. Let us note that $\psi_B(T(\Omega_i)) \le k_i \psi_{\sum_{\gamma_i} A_t}(\Omega_i)$.

From the inequality

$$\|a_i\|_{\sum_{\gamma_i} A_t} = \inf_{a_{t_j}\in A_{t_j}, t_j\in\gamma_j} \sum \|a_{t_j}\|_{A_{t_j}} \le \frac{1}{m_i}\sum_{t_j\in\gamma_j} h(t_j)\|a_{t_j}\|_{A_{t_j}} \le \frac{M}{m_i}\|a\|_A$$

we get that $\psi_{\sum_{\gamma_i} A_t}(\Omega_i) \le \frac{M}{m_i}\Omega$. Then

$$\psi_B(T(\Omega)) \le \sum_{i=1}^n k_i \frac{M}{m_i}\psi_A(\Omega) = \sum_{i=1}^n \mu_Z(\gamma_i) M = M = (1+\varepsilon)C\prod_{i=1}^n\left(\frac{k_i}{\mu_Z(\gamma_i)}\right)^{\mu_Z(\gamma_i)}.$$

Since ε is arbitrary the proof is over.

2) Let now Ω be a bounded subset of B. We use the abreviation $k_i = \beta(T_{B\to\Delta_{\gamma_i} A_t})$, $\delta = \psi_B(\Omega)$. Thus we have inequalities $\psi_{\Delta_{\gamma_i}, A_t)}(T(\Omega)) \le k_i\delta$. Let $U_1^i, \ldots, U_{s_i}^i$ be sets with diam of U^i in $\Delta_{\gamma_i} A_t$ not exceeding $k_i\delta$ and such that $t(\Omega) = \bigcup_{J=1}^{s_i} U_J^i$ and let $W_{j,k,\ldots,m} = U_j^1 \cap U_k^2 \cap U_n^m$ runs the set of all possible intersections of the sets mentioned abowe. Then $T(\Omega) \subset \Delta A_t \subset A$ because $\sup_{t\in\Gamma}\|Tx\|_{A_t} \le \sup_{t\in\Gamma}\|T\|_{B\to A_t}\|x\|_B < \infty$. For all $a, a' \in A$ the condition $A \in J(A,Z)$ gives inequality

$$\|a - a'\|_A \le C\exp\left(\int_\Gamma \log h^{-1}(t)dZ(t)\right) J(h(t), a - a', A_t),$$

where $h(t)$ is the step function admitting values $M_i = \|a - a'\|_{\Delta_{\gamma_i} A_t}^{-1}$ on the sets γ_i correspondingly. Hence

$$\|a - a'\|_A \le C\prod_{i=1}^n \|a - a'\|_{\Delta_{\gamma_i} A_t}^{\mu_Z(\gamma_i)} \sup_{t\in\Gamma} M(t)\|a - a'\|_{A_t} \le C\prod_{i=1}^n \|a - a'\|_{\Delta_{\gamma_i} A_t}^{\mu_Z(\gamma_i)}$$

and the diameter of the set $W_{j,k,\dots,m}$ in the norm of A does not exceed $C\prod_{i=1}^{n}(k_i\delta)^{\mu_Z(\gamma_i)}$. Therefore $\psi_B(T(\Omega)) \le C\prod_{i=1}^{n} k_i^{\mu_Z(\gamma_i)}\psi_A(\Omega)$ and the theorem is proved.

Since $\beta(T) = 0$ iff T is a compact operator the previous theorem is a generalization of Theorem 1 from [47], namely we have the

Corollary 2.9. *1) Let $A_t, t \in \Gamma$ be a bounded family of Banach spaces, B an arbitrary Banach space and let the Banach space A belongs to the class $K(A, Z)$. Let γ_i be a subset of Γ of positive Z-measure . Suppose that $T : \sum A_t \to B$, $\sup \|T/A_t\|_{A_t \to B_t} < \infty$ and T is a compact operator from $\sum_\gamma A_t$ into B. Then $T : A \to B$ is a compact operator.*

2) Let $A_t, t \in \Gamma$ be a bounded family of Banach spaces, B an arbitrary Banach space and let the Banach space A belongs to the class $J(A, Z)$. Suppose that $T : B \to A_t$, $\sup \|T\|_{B\to A_t} < \infty$ and T is a compact operator from B into $\Delta_\gamma A_t$. Then $T : B \to A$ is a compact operator.

If we consider a finite family $(A_1, A_2, ..., A_n)$, namely if the family A_t is constant on the sets γ_i we can replace in Theorem 2.8 $\beta\left(T_{\sum_{\gamma_i} A_t \to B}\right)$ by $\beta\left(T_{A_i\to B}\right)$ and $\beta\left(T_{B\to\Delta_{\gamma_i}A_t}\right)$ by $\beta\left(T_{B\to A_i}\right)$.

In this way we get results for interpolation of finite families and measure of noncompactness. Let us note that this type result I published in the case of Cobos-Peetre polygon method of interpolation in a much more popular paper [48]. They sound in a similar way using mentioned in the previous subsection functions $D_{\alpha,\beta}$ and $E_{\alpha,\beta}$.

Results about interpolation of some property when one of the families A_t or B_t is constant usually are called Lions-Peetre type results.

A often used approach to the case when both the families are not constant is inspired by a paper of Arne Persson from 1964, where approximation conditions appear.

To extend the results to the case when in both sides we have $n-$tuples $\overline{A}$ and $\overline{B}$ we need an approximation hypothesis on the $n-$tuple $\overline{B} = (B_1, B_2, \dots, B_n)$.

Here we work with $\tilde{\beta}(T)$.

H) There exist positive constants $C_1, C_2, \dots, C_n$ such that given any $\varepsilon > 0$ and any finite sets $F_j (F_j \subset B_j)$ there is a finite rank operator $P : B_j \to B_j$ such that $P(B_j \subset \Delta\bar{B})$, $\|I - P\|_{B_j\to B_j} \le C_j$ and $\|x - Px\|_{B_j} < \varepsilon$ for all $x \in F_j$.

The hypothesis **H** is satisfied for example by the $n-$tuple $(L^{p_2}(X, \mu), \dots, (L^{p_n}(X, \mu))$, $1 \le p_j < \infty$ where X is locally compact space with positive measure μ.

The following Lemma can be proved in a way similar to that of the lemma in [68].

Lemma 2.10. *Suppose that the Banach n-tuple $\bar{B}$ has the approximation property* ***H****. Then given any $\varepsilon > 0$ there exists $P \in L(\bar{B}, \bar{B})$ of finite rank $P(B_j) \subset \Delta\bar{B}$, such that $\|T - PT\|_{A_j \to B_j} \leq (1+\varepsilon) C_j \tilde{\beta}(T_{A_j \to B_j})$.*

I proved in [48] the following

Theorem 2.11. *Let $\bar{A}$ and $\bar{B}$ be two Banach n-tuples, $\bar{B}$ satisfying approximation hypothesis* ***H****, let (α, β) in a convex polygon Π, $1 \leq q \leq \infty$ and let 1)$A = \bar{A}_{(\alpha,\beta),q,K}$ and $B = \bar{B}_{(\alpha,\beta),q,K}$, or*
2)$A = \bar{A}_{(\alpha,\beta),q,J}$ and $B = \bar{B}_{(\alpha,\beta),q,J}$.
Then if $T \in L(\bar{A}, \bar{B})$, the following inequality holds

$$\tilde{\beta}(T_{A \to B}) \leq D_{(\alpha,\beta)} \left(C_1 \tilde{\beta}(T_{A_1 \to B_1}), C_2 \tilde{\beta}(T_{A_2 \to B_2}), \ldots, C_n \tilde{\beta}(T_{A_n \to B_n}) \right).$$

Proof. The proof uses the idea of the proof of Theorem 2 from [68]. Given $\varepsilon > 0$, let $P \in L(\bar{A}, \bar{B})$ be as in the lemma. Then $PT : A \to B$ is finite dimensional and hence $\tilde{\beta}(PT) = 0$. Let Ω be a bounded subset of A. Then
$\tilde{\psi}_B(T(\Omega)) \leq \tilde{\psi}_B(PT(\Omega)) + \tilde{\psi}_B((T - PT)(\Omega)) = \tilde{\psi}_B((T - PT)(\Omega))$
$\leq \|T - PT\|_{A \to B} \tilde{\psi}_A(\Omega)$. The last inequality follows because the bounded linear map $T - PT$ is a $\|T - PT\|_{A \to B}$ ball contraction. In view of what we have in the beginning of the subsection 3.1.3. The Dicesar function in both cases, of K-space and J-space we have

$$\|T - PT\|_{A \to B} \leq D_{(\alpha,\beta)} \left(\|T - PT\|_{A_1 \to B_1}, \|T - PT\|_{A_2 \to B_2} \ldots, \|T - PT\|_{A_n \to B_n} \right).$$

In view of the lemma

$$\tilde{\beta}(T_{A \to B}) \leq (1+\varepsilon) D_{(\alpha,\beta)} \left(C_1 \tilde{\beta}(T_{A_1 \to B_1}), C_2 \tilde{\beta}(T_{A_2 \to B_2}), \ldots, C_n \tilde{\beta}(T_{A_n \to B_n}) \right).$$

Since ε is arbitrary small we get the enquire result from the following inequality

$$\tilde{\psi}_B(T(\Omega)) \leq 1+\varepsilon) D_{(\alpha,\beta)} \left(C_1 \tilde{\beta}(T_{A_1 \to B_1}), C_2 \tilde{\beta}(T_{A_2 \to B_2}), \ldots, C_n \tilde{\beta}(T_{A_n \to B_n}) \right) \tilde{\psi}_A(\Omega).$$

□

Next we mention variants of class $K(A, Z)$:
We say that a Banach space A belongs to the class $K^S_{z_0}(\bar{A})$ if $A \subset \Sigma(\bar{A})$ and

$$\frac{K(\alpha, a)}{\alpha(z_0)} \leq C \|a\|_A,$$

for every $\alpha \in S$ and every $a \in A$.

Let $f^*(t)$, $0 < t < \infty$, denote the nonincreasing rearrangement of the function f defined on a σ-finite measure space (Ω, μ). Set $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds$ and let $\Psi(t)$ denote a measurable function (usually quasiconcave). Let now T be a C-subadditive operator from a Banach space A into a space of μ-measurable and essentially bounded functions. This means, that if $a = \sum_j a_{\gamma_j}$, with $a_{\gamma_j} \in A_{\gamma_j}$ and $A_{\gamma_j} \neq A_{\gamma_k}$ for every $j \neq k$, then

$$\left| T\left(\sum_j a_{\gamma_j}\right)(t)\right| \leq C\sum_j |T(a_{\gamma_j})(t)|$$

almost everywhere on μ.

We say that T is of weak-type (A, Ψ) (resp. weakened-type (A, Ψ)) with constant C if, for every $f \in A$ and $0 < t < \infty$,

$$(Tf)^*(t) \leq C(\|f\|_A/\Psi(t)),$$

(resp. $(Tf)^{**}(t) \leq C(\|f\|_A/\Psi(t))$).

By defining

$$L^\infty(\Psi) = \{f;\ f \text{ is measurable and } \sup_{t>0} \Psi(t)|f(t)| < \infty\},$$

we have that if $S_T f = (Tf)^{**}$, then T is of weakened type (A, Ψ) whenever $S_T : A \to L^\infty(\Psi)$.

Theorem 2.12. *Let $\bar{A}$ be an i.f., let B belongs to the class $K^S_{z_0}$ and let T be the C-subadditive operator as above.*
(a) If T is of weakened-type $(A(\gamma), \Psi_\gamma)$ with constants $c(\gamma) \in \widetilde{L}$, then $T_{|B}$ is of the weakened-type (B, Φ), where

$$\Phi(t) = \inf_{\alpha\in S} \frac{D^S_{z_0}(\alpha^{-1}c(\Psi.(t))^{-1})}{\alpha^{-1}(z_0)}$$

(b) If the i.f. is finite and T is of weak-type $(A(\gamma), \Psi_\gamma)$ with constants $c(\gamma) \in \widetilde{L}$ where the functions Ψ_γ are quasiconcave, then $T_{|B}$ is of the weak-type (B, Φ) with Φ as in (5).

Proof. (a) The proof of (a) can be easily obtained as a corollary of Theorem 4.2 and example 5.5 in [14], however we shall give a direct proof that can be used to also prove (b).

Let $a \in B$ and choose an arbitrary $\varepsilon > 0$. We can find a representation $a = \sum a_j$, $a_j \in A(\gamma_j)$, such that

$$\sum_j \alpha(\gamma_j)\|a_j\|_{\gamma_j} \leq (1+\varepsilon)K(\alpha, a)\cdot$$

By using this fact, our assumptions and the subadditive property

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t),$$

we can make the following estimates for every $\beta \in \widetilde{S}$:

$$(Ta)^{**}(t) = \Big(T(\sum_j a_j)\Big)^{**}(t) \le C\Big(\sum_j |Ta_j|\Big)^{**}(t) \le C\sum_j c(\gamma_j)\frac{\|a_j\|_{\gamma_j}}{\Psi_{\gamma_j}(t)}$$

$$\le C\alpha^{-1}(z_0)\sum_j \frac{(c(\gamma_j)\alpha^{-1}(\gamma_j)\beta(\gamma_j)}{\Psi_{\gamma_j}(t)\alpha^{-1}(z_0)}\|a_j\|_{\gamma_j}\alpha(\gamma_j)\beta^{-1}(\gamma_j)$$

$$\le C\alpha^{-1}(z_0)\beta(z_0)\sup_{\gamma\in\Gamma}\frac{\alpha(\gamma)^{-1}\beta(\gamma)c(\gamma)}{\Psi_\gamma(t)\alpha^{-1}(z_0)\beta(z_0)}\sum_j \|a_j\|_{\gamma_j}\alpha(\gamma_j)\beta^{-1}(\gamma_j)$$

$$\le C(1+\varepsilon)\frac{K(\alpha\beta^{-1},a)}{\alpha(z_0)\beta^{-1}(z_0)}\sup_{\gamma\in\Gamma}\frac{\alpha(\gamma)^{-1}\beta(\gamma)c(\gamma)}{\Psi_\gamma(t)\alpha^{-1}(z_0)\beta(z_0)}$$

$$\le C(1+\varepsilon)\|a\|_A\sup_{\gamma\in\Gamma}\frac{\alpha(\gamma)^{-1}\beta(\gamma)c(\gamma)}{\Psi_\gamma(t)\alpha^{-1}(z_0)\beta(z_0)}\cdot$$

Thus, by taking infimum over all $\beta \in \widetilde{S}$ first and then over all $\alpha \in S$, we find that

$$(Ta)^{**}(t) \le C\|a\|_A \inf_{\alpha\in S}\frac{D^S_{z_0}(\alpha^{-1}c(\Psi_{\cdot}(t))^{-1})}{\alpha^{-1}(z_0)}\cdot$$

(b) The proof is similar to that of (a). In fact, we only need to replace the corresponding estimates in the proof of (a) by the following estimates:

$$\Big(\sum_{j=1}^N |Ta_j(t)|\Big)^* \le \sum_{j=1}^N (Ta_j)^*(t/n) \le \sum_j^n c(\gamma_j)\|a_j\|_{A(\gamma_j)}/\Psi_{\gamma_j}(t/n)$$

$$\le n\sum_j^n c(\gamma_j)\|a_j\|_{A(\gamma_j)}/\Psi_{\gamma_j}(t)\cdot$$

These estimates hold because $(f+g)^*(t_1+t_2)^{\le} f^*(t_1)+g^*(t_2)$ and $\Psi_{\gamma_j}(t)/t$ are nonincreasing so that, in particular, $\Psi_{\gamma_j}(t/n) \ge \Psi_{\gamma_j}(t)/n$. □

By using this theorem (and some results from [14]) we obtain the following result for the complex (St. Louis) spaces $A[z_0]$:

Corollary 2.13. *Let $A = A[z_0]$ ($z_0 \in D$) be the complex interpolation space for the i.f. $\bar{A}$ and let $\Psi_\gamma(t)$ be measurable functions in $\widetilde{L}$, for every fixed t. If $T_{|A(\gamma)}$ is of weakened type $(A(\gamma), \Psi_\gamma)$ with constants $c(\gamma) \in \widetilde{L}$, then $T_{|A[z_0]}$ is of weakened type $(A[z_0], \Psi_{z_0})$.*

Proof. According to Theorem 2.1 in [14] and well-known interpolation theorems for the complex method (see [19]) we have that $A[z_0] \in K^S_{z_0}$ for any group S suitable for the constructions of the real method. Therefore, the proof follows by applying the above theorem. □

2.3. **Triples of Banach function spaces.** Here the results are from the paper [3], written by 5 authors. It is a long paper - 36 pages. I will comment some of the results from there. My personal contribution to this work is mainly in the proof of second reiteration formula and in the interpolation of block-Lorentz spaces, so I will present here the proofs of this part of the results.

Let X_0, X_1, X_2 be a triple of Banach or quasi-Banach triple.

For $t_1, t_2 > 0$ and $x \in X_0 + X_1 + X_2$ the $K-$functional has the form:

$$K(t_1, t_2, x; \bar{X}) = \inf\{(\|x_0\|_{X_0} + t_1 \|x_1\|_{X_1} + t_2 \|x_2\|_{X_2}),\quad x = x_0 + x_1 + x_2, x_0 \in X_0, x_1 \in X_1, x_2 \in X_2\}.$$

Let $H = \{(\theta_1, \theta_2) : \theta_1 > 0,\ \theta_2 > 0,\ \ \theta_1 + \theta_2 < 1\}$

The space $\bar{X}_{\vec{\theta},q,K} = \bar{X}_{(\theta_1,\theta_2),q,K}$ is defined for $0 < q \le \infty$ and $\vec{\theta} = (\theta_1, \theta_2) \in H$, as the set of all $x \in X_0 + X_1 + X_2$ for which the norm (or quasinorm, for simplicity we always say norm)

$$\|x\|_{\vec{\theta},q,K} = \left(\int_0^\infty \int_0^\infty (t_1^{-\theta_1} t_2^{-\theta_2} K(t_1, t_2, x; \bar{X}))^q \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{\frac{1}{q}} \tag{2.3}$$

is finite and with the usual modification for $q = \infty$.

On the other hand, let $\bar{X}_{\vec{\theta},q,J} = \bar{X}_{(\theta_1,\theta_2),q,J}$ be the space of all those elements $x \in X_0 + X_1 + X_2$ which can be represented in form

$$x = \int_0^\infty \int_0^\infty u(t_1, t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2} \quad \text{(convergence in } X_0 + X_1 + X_2),$$

where $u(t_1, t_2)$ is strongly measurable $X_0 \cap X_1 \cap X_2-$valued function and satisfies

$$\left(\int_0^\infty \int_0^\infty (t_1^{-\theta_1} t_2^{-\theta_2} J(t_1, t_2, u(t_1, t_2); \bar{X}))^q \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{\frac{1}{q}} < \infty$$

with $J(t_1, t_2, u; \bar{X}) = \max(\|u\|_{X_0}, \|u\|_{X_1}, \|u\|_{X_2})$.

The norm $\|x\|_{\vec{\theta},q,J}$ in $\bar{X}_{\vec{\theta},q,J}$ is the infimum of the integral above over all representations of x like double integral of $u(t_1, t_2)$.

We have to mention that in contrast to the classical case of Banach or quasi-Banach couples, where K- and J- methods coincide to within equivalence of quasi-norms (i.e. $\bar{X}_{\vec{\theta},q,K} = \bar{X}_{\vec{\theta},q,J}$, this is usualy called equivalence theorem for the K- and J- methods), it is not true in general that $\bar{X}_{\vec{\theta},q,K}$ coincides with $\bar{X}_{\vec{\theta},q,J}$ for more than two spaces.

2.3.1. *On reiteration theorems in case of triples.* In 1964 Lions and Peetre proved one of the most important theoretical in interpolation theory, the so called reiteration formula for couples of Banach spaces:

$$(\bar{X}_{\theta_0,q_0}, \bar{X}_{\theta_1,q_1})_{\lambda,q} = \bar{X}_{\theta,q}, \quad \theta = (1-\lambda)\theta_0 + \lambda\theta_1$$

where $\theta_0 \neq \lambda\theta_1$.

This formula also holds for quasi-Banach spaces and as we mentioned gives a possibility to calculate the interpolation spaces for rather complicated couples $(\bar{X}_{\theta_0,q_0}, \bar{X}_{\theta_1,q_1})$ by using simpler initial couple $\bar{X} = (X_0, X_1)$.

The classical proof of the reiteration formula is based on the equivalence theorem for the K- and J- methods, which is valid for any couple of quasi-Banach spaces([6] Th.3.11.3.) G.Sparr, who defined the K- and J- methods for n-tuples, showed that we always have the imbedding $\bar{X}_{\vec{\theta},q,J} \subset \bar{X}_{\vec{\theta},q,K}$. Moreover Sparr tried to extend the reiteration formula to n-tuples and he showed that if an anlogue of the equivalence theorem is valid for the triple $\bar{X} = (X_0, X_1, X_2)$ then the anlogue of the Lions-Peetre reiteration formula is also true. But there are troubles with the equivalence theorem, when $n > 2$. Yoshikawa(1970) and Sparr(1974) gave counterexamples to the equivalence theorem, which means that the above imbedding is strict for $n > 2$. Their counterexamples could be considered somewhat "artificial" since it has the property that the intersection of the spaces is just $\{0\}$. In 1987 Cwikel and Janson [21] constructed a "non-degenerate" 3-tuple $\bar{H} = (H_0, H_1, H_2)$ of Hilbert spaces with the intersection $H_0 \cap H_1 \cap H_2$ dense in each H_i and such that embedding $\bar{H}_{\vec{\theta},q,J} \subset \bar{H}_{\vec{\theta},q,K}$ is strict, i.e. the equivalence theorem is not valid. Thus we conclude that even for a good triple, such as a triple of Hilbert spces, the classical method of proving the reiteration theorem does not work. An example, given by I.Asekritova shows, that using another way to prove reiteration theorem , which works for couples, namely so called K-divisibility, can not be applied for triples, because K-dividibility is not valid for triples. In this situation, that even for simple triples the situation was not clear it was natural to try to prove reiteration theorem for wide and important classes of triples and to find interesting applications. In 1997 Asekritova and Krugljak showed that equivalence theorem is in fact valid for any n-tuple of Banach function lattices on Ω. The proof is rather complicated and uses significantly the structure of Banach function lattices, moreover from their investigations it can be seen that it is enough to consider quasi-Banach function lattises on Ω. Using their result and Sparr's theorem 9.1 [66] we can derive the following *first reiteration theorem* for n-tuples, but for our purpose it is enough to have the simpliest case of triples):

Theorem 2.14. *(First Reiteration theorem) Let $\bar{X} = (X_0, X_1, X_2)$ be a triple of Banach or quasi-Banach function lattices on Ω and let $\bar{\lambda} = (\lambda_1, \lambda_2) \in H, \theta_i = \theta_1^i, \theta_2^i) \in H, i = 0, 1, 2$. Then*

$$(\bar{X}_{\theta_0,q_0}, \bar{X}_{\theta_1,q_1}, \bar{X}_{\theta_2,q_2})_{\lambda,q} = \bar{X}_{\theta,q}, \quad \bar{\theta} = (1 - \lambda_1 - \lambda_2)\bar{\theta}_0 + \lambda_1\bar{\theta}_1 + \lambda_2\bar{\theta}_2$$

, whenever the vectors $\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2$ are not colinear.

Note that the last condition is just anologue of the necessary condition $\theta_0 \neq \theta_1$ for the reiteration formula for couples to hold.

Theorem 2.15. *(Second Reiteration theorem) Let* $\bar{X} = (X_0, X_1, X_2)$ *be a triple of Banach or quasi-Banach function lattices on* Ω*. If* $0 < q_0, q_1, q < \infty$ *and* $\frac{1}{q} = \frac{1-\mu}{q_0} + \frac{\mu}{q_1}$*, then*

$$((X_0, X_2)_{\alpha_0, q_0}, (X_1, X_2)_{\alpha_1, q_1})_{\mu, q} = (X_0, X_1, X_2)_{(\theta_1, \theta_2), q},$$

$$\textit{where } \theta_1 = (1-\alpha_1)\mu, \quad \theta_2 = \alpha_0(1-\mu) + \alpha_1\mu.$$

First we prove the following lemma of independet interest:

Lemma 2.16. *Let* $\bar{X} = (X_0, X_1, X_2)$ *be a triple of quasi-Banach spaces.If* $0 < \alpha_0, \alpha_1, \mu < 1$ *then*

$$(a)\ (X_0, X_1, X_2)_{(\theta_1, \theta_2), 1, K} \supset ((X_0, X_2)_{\alpha_0, 1, K}, (X_1, X_2)_{\alpha_1, 1, K})_{\mu, 1, K}$$

and

$$(b)\ (X_0, X_1, X_2)_{(\theta_1, \theta_2), 1, J} \subset ((X_0, X_2)_{\alpha_0, 1, J}, (X_1, X_2)_{\alpha_1, 1, J})_{\mu, 1, J}$$

$$\textit{where } \theta_1 = (1-\alpha_1)\mu, \quad \theta_2 = \alpha_0(1-\mu) + \alpha_1\mu.$$

Proof. (a) Let $Y_i = (X_i, X_2)_{\alpha_i, 1, K}, i + 1, 2$. For any $\varepsilon > 0$ we take an "almost optimal decomposition" $f = f_0 + f_1$ with $f_0 \in Y_0, f_1 \in Y_1$ and

$$\|f_0\|_{Y_0} + t\|f_1\|_{Y_1} \leq (1+\varepsilon)K(t, f : Y_0, Y_1).$$

Note that

$$K(t_1, t_2, f; X_0, X_1, X_2) \leq C[K(t_2, f_0; X_0, X_2) + t_1 K(t_2/t_1, f_1; X_1, X_2)].$$

Indeed if we take almost optimal decomposition $f_0 = f_0^0 + f_2^0$ and $f_1 = f_1^1 + f_2^1$ such that

$$\|f_0^0\|_{X_0} + t_2\|f_2^0\|_{X_2} \leq (1+\varepsilon)K(t_2, f_0 : X_0, X_2)$$

and

$$\|f_1^1\|_{X_1} + \frac{t_2}{t_1}\|f_2^1\|_{X_2} \leq (1+\varepsilon)K(\frac{t_2}{t_1}, f_1 : X_1, X_2)$$

then $f_0^0 + f_2^0 + f_1^1 + f_2^1 = f_0 + f_1 = f$ and hence

$$\begin{aligned} K(t_1, t_2, f; X_0, X_1, X_2) &\leq \|f_0^0\|_{X_0} + t_1\|f_1^1\|_{X_1} + t_2\|f_2^0 + f_2^1\|_{X_2} \\ &\leq \|f_0^0\|_{X_0} + Ct_2\|f_2^0\|_{X_2} + t_1\|f_1^1\|_{X_1} + Ct_2\|f_2^1\|_{X_2} \\ &\leq C(1+\varepsilon)K(t_2, f_0; X_0, X_2) + C(1+\varepsilon)t_1 K(t_2/t_1, f_1; X_1, X_2). \end{aligned}$$

Since ε was arbitrary we obtain the wanted inequality.

By using it we get

$$\|f\|_{(\theta_1, \theta_2), 1, K} = \int_0^\infty \int_0^\infty t_1^{-\theta_1} t_2^{-\theta_2} K(t_1, t_2, f; \bar{X}) \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

$$\leq C\int_0^\infty\int_0^\infty t_1^{-\theta_1}t_2^{-\theta_2}[K(t_2,f_0;X_0,X_2)+t_1K(t_2/t_1,f_1;X_1,X_2)]\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

$$=C\int_0^\infty\int_0^\infty (t_1^{1-\alpha_1}t_2^{\alpha_1-\alpha_0})^{-\mu}[t_2^{-\alpha_0}K(t_2,f_0;X_0,X_2)]\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

$$+C\int_0^\infty\int_0^\infty (t_1^{1-\alpha_1}t_2^{\alpha_1-\alpha_0})^{1-\mu}[(t_2/t_1)^{-\alpha_1}K(t_2/t_1,f_1;X_1,X_2)]\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

$$=C(I_1+I_2).$$

In the first integral we make the change of variables

$$\tau_1=t_1^{1-\alpha_1}t_2^{\alpha_1-\alpha_0},\quad \tau_2=t_2$$

with Jacobian $J(\tau_1,\tau_2)=(1-\alpha_1)^{-1}\tau_1^{\alpha_1/(1-\alpha_1)}\tau_2^{-(\alpha_1-\alpha_0)/(1-\alpha_1)}$ and hence

$$I_1=(1-\alpha_1)^{-1}\int_0^\infty\int_0^\infty \tau_1^{-\mu}\tau_2^{-\alpha_0}K(\tau_2,f_0;X_0,X_2)\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}$$

$$=(1-\alpha_1)^{-1}\int_0^\infty \tau_1^{-\mu}\|f_0\|_{(X_0,X_2)_{\alpha_0,1,K}}\frac{d\tau_1}{\tau_1}.$$

In the second integral we make the change of variables

$$\tau_1=t_1^{1-\alpha_1}t_2^{\alpha_1-\alpha_0},\quad \tau_2=t_2/t_1$$

with Jacobian $J(\tau_1,\tau_2)=(1-\alpha_0)^{-1}\tau_1^{(1+\alpha_0)/(1-\alpha_0)}\tau_2^{2(\alpha_0-\alpha_1)/(1-\alpha_0)}$ which gives

$$I_2=(1-\alpha_0)^{-1}\int_0^\infty\int_0^\infty \tau_1^{(1-\mu)}\tau_2^{-\alpha_1}K(\tau_2,f_1;X_1,X_2)]\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}$$

$$=(1-\alpha_0)^{-1}\int_0^\infty \tau_1^{(1-\mu)}\|f_1\|_{(X_1,X_2)_{\alpha_1,1,K}}\frac{d\tau_1}{\tau_1}.$$

Thus, setting $\gamma=C\max((1-\alpha_1)^{-1},(1-\alpha_0)^{-1})$ we have

$$\|f\|_{(\theta_1,\theta_2),1,K}\leq\gamma\int_0^\infty \tau_1^{-\mu}[\|f_0\|_{(X_0,X_2)_{\alpha_0,1,K}}+\tau_1\|f_1\|_{(X_1,X_2)_{\alpha_1,1,K}}]\frac{d\tau_1}{\tau_1}$$

$$=\gamma\int_0^\infty \tau_1^{-\mu}[\|f_0\|_{(Y_0}+\tau_1\|f_1\|_{Y_1}]\frac{d\tau_1}{\tau_1}$$

$$\leq\gamma(1+\varepsilon)\int_0^\infty \tau_1^{-\mu}K(\tau_1,f;Y_0,Y_1)\frac{d\tau_1}{\tau_1}$$

which shows the required embedding.

(b) Let $Z_i=(X_i,X_2)_{\alpha_i,1,J}, i+1,2$. Assume $f\in(X_0,X_1,X_2)_{(\theta_1,\theta_2),1,J}$. Then f can be represented in the form

$$f=\int_0^\infty\int_0^\infty u(t_1,t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\quad\text{(convergence in } X_0+X_1+X_2),$$

where $u(t_1,t_2)$ is strongly measurable $X_0\cap X_1\cap X_2-$valued function and satisfies

$$\left(\int_0^\infty \int_0^\infty (t_1^{-\theta_1} t_2^{-\theta_2} J(t_1, t_2, u(t_1, t_2); \bar{X}))^q \frac{dt_1}{t_1} \frac{dt_2}{t_2}\right)^{\frac{1}{q}} \leq (1+\varepsilon)\|f\|_{(\theta_1,\theta_2),1,J}.$$

Put

$$v(t_1) = \frac{1}{1-\alpha_1} \int_0^\infty u(t_1^{1/(1-\alpha_1)} t_2^{\alpha_0-\alpha_1/(1-\alpha_1)}, t_2) \frac{dt_2}{t_2}$$

Then $\qquad f = \int_0^\infty v(t_1) \frac{dt_1}{t_1}.$

In fact, the change of variables

$$\tau_1 = t_1^{1/(1-\alpha_1)} t_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)}, \quad \tau_2 = t_2$$

has Jacobian $J(\tau_1, \tau_2) = (1-\alpha_1)\tau_1^{-\alpha_1} \tau_2^{\alpha_1-\alpha_0}$ and hence

$$\int_0^\infty v(t_1) \frac{dt_1}{t_1} = \frac{1}{1-\alpha_1} \int_0^\infty \int_0^\infty u(t_1^{1/(1-\alpha_1)} t_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)}, t_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2} =$$

$$= \int_0^\infty \int_0^\infty u(\tau_1, \tau_2) \frac{d\tau_1}{\tau_1} \frac{d\tau_2}{\tau_2} = f$$

Hence

$$\|f\|_{(Z_0,Z_1)_{\mu,1;J}} \leq \int_0^\infty t_1^{-\mu} J(t_1, v(t_1); Z_0, Z_1) \frac{dt_1}{t_1}$$

$$\leq \int_0^\infty t_1^{-\mu} \|v(t_1)\|_{Z_0} \frac{dt_1}{t_1} + \int_0^\infty t_1^{1-\mu} \|v(t_1)\|_{Z_1} \frac{dt_1}{t_1} = J_1 + J_2.$$

To estimate the first integral we use the definition of the norm in $Z_0 = (X_0, X_1)_{\alpha_0,1;J}$ and from representation of $v_1(t)$ we get

$$J_1 = \int_0^\infty t_1^{-\mu} \|v(t_1)\|_{Z_0} \frac{dt_1}{t_1}$$

$$\leq \frac{1}{1-\alpha_1} \int_0^\infty \int_0^\infty t_1^{-\mu} t_2^{-\alpha_0} J(t_2, u(t_1^{1/(1-\alpha_1)} t_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)}, t_2); X_0, X_2) \frac{dt_2}{t_2} \frac{dt_1}{t_1}$$

$$\leq \frac{1}{1-\alpha_1} \int_0^\infty \int_0^\infty t_1^{-\mu} t_2^{-\alpha_0} [\|u(t_1^{1/(1-\alpha_1)} t_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)}, t_2)\|_{X_0}$$

$$+ t_2 \|u(t_1^{1/(1-\alpha_1)} t_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)}, t_2)\|_{X_2}] \frac{dt_2}{t_2} \frac{dt_1}{t_1}.$$

Changing variables

$$\tau_1 = t_1^{1/(1-\alpha_1)} t_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)}, \quad \tau_2 = t_2$$

with Jacobian $J(\tau_1, \tau_2) = (1-\alpha_1)\tau_1^{-\alpha_1} \tau_2^{\alpha_1-\alpha_0}$, we obtain

$$J_1 \leq \int_0^\infty \int_0^\infty \tau_1^{-\theta_1} \tau_2^{-\theta_2} [\|u(\tau_1, \tau_2)\|_{X_0} + \tau_2 \|u(\tau_1, \tau_2)\|_{X_2}] \frac{d\tau_2}{\tau_2} \frac{d\tau_1}{\tau_1}$$

$$\le 2\int_0^\infty\int_0^\infty \tau_1^{-\theta_1}\tau_2^{-\theta_2}J(\tau_1,\tau_2,u(\tau_1,\tau_2);X_0,X_1,X_2)\frac{d\tau_2}{\tau_2}\frac{d\tau_1}{\tau_1}$$
$$\le 2(1+\varepsilon)\|f\|_{(\theta_1,\theta_2),1;J}.$$

The second integral can be estimated similarly. We put here the estimates just for completness. First we rewrite the representation about $v(t_1)$ in the form

$$v(t_1)=\frac{1}{1-\alpha_1}\int_0^\infty u(t_1^{1/(1-\alpha_1)}s_2^{(\alpha_0-\alpha_1)/(1-\alpha_1)},s_2)\frac{ds_2}{s_2}$$
$$=\frac{1}{1-\alpha_0}\int_0^\infty u(t_1^{1/(1-\alpha_0)}t_2^{(\alpha_0-\alpha_1)/(1-\alpha_0)},t_1^{1/(1-\alpha_0)}t_2^{(1-\alpha_1)/(1-\alpha_0)})\frac{dt_2}{t_2}$$

and then

$$J_2=\int_0^\infty t_1^{(1-\mu)}\|v(t_1)\|_{Z_1}\frac{dt_1}{t_1}$$
$$\le\frac{1}{1-\alpha_0}\int_0^\infty\int_0^\infty t_1^{(1-\mu)}t_2^{-\alpha_1}J(t_2,u(t_1^{1/(1-\alpha_0)}t_2^{(\alpha_0-\alpha_1)/(1-\alpha_0)},t_1^{1/(1-\alpha_0)}t_2^{(1-\alpha_1)/(1-\alpha_0)});X_1,X_2)\frac{dt_2}{t_2}\frac{dt_1}{t_1}$$
$$\le\frac{1}{1-\alpha_0}\int_0^\infty\int_0^\infty t_1^{(1-\mu)}t_2^{-\alpha_1}[\|u(t_1^{1/(1-\alpha_0)}t_2^{(\alpha_0-\alpha_1)/(1-\alpha_0)},t_1^{1/(1-\alpha_0)}t_2^{(1-\alpha_1)/(1-\alpha_0)})\|_{X_1}$$
$$+t_2\|u(t_1^{1/(1-\alpha_0)}t_2^{(\alpha_0-\alpha_1)/(1-\alpha_0)},t_1^{1/(1-\alpha_0)}t_2^{(1-\alpha_1)/(1-\alpha_0)})\|_{X_2}]\frac{dt_2}{t_2}\frac{dt_1}{t_1}.$$

Changing variables

$$\tau_1=t_1^{1/(1-\alpha_0)}t_2^{(\alpha_0-\alpha_1)/(1-\alpha_0)},\quad \tau_2=t_1^{1/(1-\alpha_0)}t_2^{(1-\alpha_1)/(1-\alpha_0)}$$

with Jacobian $J(\tau_1,\tau_2)=(1-\alpha_0)\tau_1^{-\alpha_1}\tau_2^{\alpha_1-\alpha_0}$, we obtain

$$J_2\le\int_0^\infty\int_0^\infty \tau_1^{-\theta_1}\tau_2^{-\theta_2}\tau_1[\|u(\tau_1,\tau_2)\|_{X_1}+\frac{\tau_2}{\tau_1}\|u(\tau_1,\tau_2)\|_{X_2}]\frac{d\tau_2}{\tau_2}\frac{d\tau_1}{\tau_1}$$
$$\le 2\int_0^\infty\int_0^\infty \tau_1^{-\theta_1}\tau_2^{-\theta_2}J(\tau_1,\tau_2,u(\tau_1,\tau_2);X_0,X_1,X_2)\frac{d\tau_2}{\tau_2}\frac{d\tau_1}{\tau_1}$$
$$\le 2(1+\varepsilon)\|f\|_{(\theta_1,\theta_2),1;J}.$$

By putting together all the above estimates we obtain

$$\|f\|_{(Z_0,Z_1)_{\mu,1;J}}\le 4(1+\varepsilon)\|f\|_{(\theta_1,\theta_2),1;J}$$

and the Lemma is proved. □

Proof of Theorem 2.15.

By using the power theorem for quasi-Banach couples ([6], Th.3.11.6) we have

$$[((X_0,X_2)_{\alpha_0,q_0},(X_1,X_2)_{\alpha_1,q_1})_{\mu,q}]^q=([(X_0,X_2)_{\alpha_0,q_0}]^{q_0},[(X_1,X_2)_{\alpha_1,q_1}]^{q_1})_{\eta,1},$$

where $\eta=\mu q/q_1$ or equivalently $1-\eta=(1-\mu)q/q_0$.

We can find $0 < \beta_0, \beta_1 < 1$ such that $s_2 = q_0\alpha_0/\beta_0 = q_1\alpha_1/\beta_1$ and we also put $s_0 = q_0(1-\alpha_0)/(1-\beta_0)$ and $s_1 = q_1(1-\alpha_1)/(1-\beta_1)$. We can do this because if for example $q_1\alpha_1 \geq q_0\alpha_0$ then we choose $\beta_1 \in [\beta_0, 1)$ and take $\beta_0 = \beta_1 q_0\alpha_0/(q_1\alpha_1)$. We see that $\beta_0 < 1$. Again by the power theorem for quasi-Banach couples we find that

$$([(X_0, X_2)_{\alpha_0,q_0}]^{q_0}, [(X_1, X_2)_{\alpha_1,q_1}]^{q_1})_{\eta,1} = ((X_0^{s_0}, X_2^{s_2})_{\beta_0,1}, (X_1^{s_1}, X_2^{s_2})_{\beta_1,1})_{\mu,1}$$

By Lemma 2.16 and equivalence Theorem 1 of [2] for quasi-Banach function lattices we find that

$$((X_0^{s_0}, X_2^{s_2})_{\beta_0,1}, (X_1^{s_1}, X_2^{s_2})_{\beta_1,1})_{\mu,1} = (X_0^{s_0}, X_1^{s_1}, X_2^{s_2})_{(\lambda_1,\lambda_2),1},$$

where

$$\lambda_1 = (1-\beta_1)\eta \text{ and } \lambda_2 = \beta_0(1-\eta) + \beta_1\eta.$$

By using the power theorem of Sparr [[66], Th.7.1] for triples we obtain

$$(X_0^{s_0}, X_1^{s_1}, X_2^{s_2})_{(\lambda_1,\lambda_2),1} = [(X_0, X_1, X_2)_{(\lambda_1,\lambda_2),1}]^q$$

where

$$\theta_1 = \lambda_1 s_1/q,\ \theta_2 = \lambda_2 s_2/q, \quad \text{and } (1-\theta_1-\theta_2) = (1-\lambda_1-\lambda_2)s_0/q.$$

Thus

$$((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_2)_{\alpha_1,q_1})_{\mu,q} = (X_0, X_1, X_2)_{(\theta_1,\theta_2),q},$$

with equivalent quasi-norms, note that $\theta_1 = \lambda_1 s_1/q = (1-\alpha_1)\mu$, $\quad \theta_2 = \lambda_2 s_2/q = \alpha_0(1-\mu) + \alpha_1\mu$, and this ends the proof. □

Remark 2.17. For the case of $q_0 = q_1, \alpha_0 = \alpha_1$ and for a triple of Banach spaces for which the equivalence theorem holds, the result of Theorem 2.15 was also pointed by Yoshikawa [70], Prop..1,.2 (see also Sparr [66], Th.3 p. 292, and in the particular case $n = 2, m = 1$, p.291).

Remark 2.18. In the case when $\alpha_1 = 0$, i.e. when instead of the space $(X_1, X_2)_{\alpha_1,q_1}$ we have the space X_1, then the formula from the Theorem 2.15 is not true in general. In fact, there is an example, given in [3], where a triple of weighted L_p plays the role of the spaces X_i. In this example

$$((X_0, X_2)_{\alpha_0,q_0}, X_1)_{\mu,q} \neq (X_0, X_1, X_2)_{(\mu,\alpha_0(1-\mu)),q}.$$

Note that in Theorem 2.15 the second space in the couples (X_0, X_2) and (X_1, X_2) is the same space, and this is essential since the formula

$$((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_3)_{\alpha_1,q_1})_{\mu,q} = (X_0, X_1, X_2, X_3)_{((1-\alpha_1)\mu,\alpha_0(1-\mu),\alpha_1\mu),q}$$

is not true in general. Indeed, if we take $X_1 = X_3$, then

$$(X_0, X_1, X_2, X_3)_{((1-\alpha_1)\mu,\alpha_0(1-\mu),\alpha_1\mu),q} = (X_0, X_1, X_2)_{(\mu,\alpha_0(1-\mu)),q}$$

and as noted before, this is different from

$$((X_0, X_2)_{\alpha_0,q_0}, X_1)_{\mu,q} = ((X_0, X_2)_{\alpha_0,q_0}, (X_1, X_3)_{\alpha_1,q_1})_{\mu,q}.$$

Theorem 2.15 also gives the possibility to calculate $(X_0, X_1, X_2)_{(\theta_1,\theta_2),q}$ if we can calculate the interpolation spaces for some couples.

Corollary 2.19. *Let (X_0, X_1, X_2) be a triple of Banach or quasi-Banach function lattices on Ω and $0 < q < \infty$. Then*

$$(X_0, X_1, X_2)_{(\theta_1,\theta_2),q} = ((X_0, X_2)_{\theta_2,q}, (X_1, X_3)_{\theta_2,q})_{\theta,q}$$

where $\theta = \theta_1/(1-\theta_2)$.

This follows immediately from Theorem 2.15 by putting $\alpha_0 = \alpha_1 = \theta_2$, $q_1 = q_2 = q$ and $\mu = \theta_1/(1-\theta_2)$.

2.3.2. *Interpolation of a family of block-Lorentz spaces.*

Let ω be a weight function on Ω and consider the weighted space $L_p^\sigma = L_p^\sigma(\omega^\sigma)$, for $\sigma \in \mathbb{R}$ and $0 \leq p \leq \infty$, on Ω defined by the finiteness ot the quasi-norms $\|f\|_{L_p^\sigma} = \left(\int_\Omega |f(x)\omega(x)^\sigma|^p \, d\mu\right)^{1/p}$. Then according to the Stein-Weiss interpolation theorem (see [6], Th. 5.5.1), in the diagonal case, i.e., when $1/p = (1-\theta)/p_0 + \theta/p_1$, we have

$$(L_{p_0}^{\sigma_0}, L_{p_1}^{\sigma_1})_{\theta,p} = L_p^\sigma, \quad \sigma = (1-\theta)\sigma_0 + \theta\sigma_1.$$

If we now indentify the spaces $L_{p_i}^{\sigma_i}$ with the points $(\sigma_i, 1/p_i)$, $i = 0, 1$, in the upper half plane, then the interpolation space L_p^σ will be identified with the point $(\sigma, 1/p)$ on the straight line between $(\sigma_1, 1/p_1)$ and $(\sigma_2, 1/p_2)$.

Let us now introduce the sets

$$\Omega_k = \Omega_k(\omega) = \left\{x \in \Omega : 2^k \leq \omega(x) \leq 2^{k+1}\right\}, \quad k \in \mathbb{Z},$$

and define, for $\sigma \in \mathbb{R}$ and $0 \leq p, q, r, \leq \infty$, the *block - Loretntz spaces* $L_{p,r}^{\sigma,q} = L_{p,r}^{\sigma,q}(\omega)$ by the finiteness of the quasi-norm

$$\|f\|_{L_{p,r}^{\sigma,q}} = \left(\sum_{k\in\mathbb{Z}} (\|f\omega^\sigma \chi_{\Omega_k}\|_{L_{p,r}})^q\right)^{1/q}$$

with the standard modification for $q = \infty$. Here, $L_{p,r}$ denotes the usual Lorentz space (with the convention that $r = \infty$ when $p = \infty$). It is clear that $L_{p,p}^{\sigma,p} = L_p^\sigma$ and in what follows we also use the notation $L_p^{\sigma,q} := L_{p,p}^{\sigma,q}$.

In the case when $r = p$ and $\omega(x) = |x|$ on $\mathbb{R}^n \setminus \{0\}$, the spaces $L^{\sigma,q}_{p,r}$ are the so called homogenous Herz spaces $K^{\sigma,q}_p$.

According to the next lemma (due to Gilbert [29], Th. 3.7), for $p_0 = p_1 = p$ we can calculate the interpolation spaces $(L^{\sigma_0}_{p_0}, L^{\sigma_1}_{p_1})_{\theta,q}$ even in the non-diagonal case.

Lemma 2.20. *If $\sigma_0 \neq \sigma_1$, then*

$$(L^{\sigma_0}_{p_0}, L^{\sigma_1}_{p_1})_{\theta,q} = L^{\sigma,q}_p, \ \text{ where } \sigma = (1-\theta)\sigma_0 + \theta\sigma_1.$$

Proof. For the the reader's convenience we give another and, in our oppinion clearer proof than in [29]. It is enough to consider the case $\sigma_1 > \sigma_0$. The other case is obtained by just interchanging the spaces.

Set

$$a_k = \left(\int_{\Omega_k} |f(x)\omega(x)^{\sigma_0}|^p \, d\mu\right)^{1/p}, \qquad k \in \mathbb{Z}.$$

According to the definition of Ω_k we have the equivalence

$$\omega(x)^{\sigma_1-\sigma_0} \approx 2^{k(\sigma_1-\sigma_0)} \quad \text{for } x \in \Omega_k, \tag{2.4}$$

or more precisely

$$2^{k(\sigma_1-\sigma_0)} \leq \omega(x)^{\sigma_1-\sigma_0} \leq 2^{(k+1)(\sigma_1-\sigma_0)} \leq \omega(x) \quad \text{for } x \in \Omega_k.$$

By using these estimates we find that

$$\begin{aligned}
\|f\|_{L^{\sigma_1}_p} &= \left(\sum_{k\in\mathbb{Z}} (|f(x)\omega(x)^{\sigma_0}|^p \omega(x)^{(\sigma_1-\sigma_0)p} \, d\mu\right)^{1/p} \\
&\leq \left(\sum_{k\in\mathbb{Z}} (2^{k(\sigma_1-\sigma_0)} a_k)^p\right)^{1/p} = \|\{a_k\}\|_{L^{\sigma_1-\sigma_0}_p} \quad \text{and}
\end{aligned}$$

$$\|f\|_{L^{\sigma_1}_p} \leq 2^{\sigma_1-\sigma_0} \|\{a_k\}\|_{L^{\sigma_1-\sigma_0}_p}, \quad \text{where}$$

$$\|\{a_k\}\|_{L^{\sigma_1-\sigma_0}_p} = \left(\sum_{k\in\mathbb{Z}} (2^{ks}|a_k|)^p\right)^{1/p}, \quad \text{for } s \in \mathbb{R} \text{ and } 0 < p \leq \infty.$$

The above estimates give

$$K(t, f; L^{\sigma_0}_p, L^{\sigma_1}_p) \approx K(t, \{a_k\}; l_p, l^{\sigma_1-\sigma_0}_p). \tag{2.5}$$

In fact, if $f = f_0 + f_1$ with $f_i \in L^{\sigma_i}_p$ $(i = 0, 1)$, then for

$$a_{k,i} = (|f_i(x)\omega(x)^{\sigma_0}|^p \, d\mu)^{1/p},$$

we have $\{a_{k,0}\} \in l^{\sigma_1-\sigma_0}_p$ and $a_k \leq 2^{\max(0,1/p-1)}(a_{k,0} - a_{k,1})$ for all $k \in \mathbb{Z}$. Hence

$$\begin{aligned}
K(t, \{a_k\}; l_p, l^{\sigma_1-\sigma_0}_p) &\leq 2^{\max(0,1/p-1)}(\|\{a_{k,0}\}\|_{l_p} + t\|\{a_{k,1}\}\|_{l^{\sigma_1-\sigma_0}_p}) \\
&\leq 2^{\max(0,1/p-1)}(\|f_0\|_{L^{\sigma_0}_p} + t\|f_1\|_{L^{\sigma_1}_p})
\end{aligned}$$

and so

$$K(t,\{a_k\};l_p,l_p^{\sigma_1-\sigma_0}) \le 2^{\max(0,1/p-1)} K(t,f;L_p^{\sigma_0},L_p^{\sigma_1}).$$

On the other hand, if $a_k = b_k + c_k$ with $\{b_k\} \in l_p$ and $\{b_k\} \in l_p^{\sigma_1-\sigma_0}$, then by putting

$$f\chi_{\Omega_k} = f\chi_{\Omega_k}\frac{b_k}{a_k} + f\chi_{\Omega_k}\frac{c_k}{a_k} \quad \text{when } a_k \neq 0$$

we obtain

$$\begin{aligned}
K(t,f;L_p^{\sigma_0},L_p^{\sigma_1}) &\le \left\| \sum_{k\in\mathbb{Z}} f\chi_{\Omega_k}\frac{b_k}{a_k} \right\|_{L_p^{\sigma_0}} + t\left\| \sum_{k\in\mathbb{Z}} f\chi_{\Omega_k}\frac{c_k}{a_k} \right\|_{L_p^{\sigma_1}} \\
&= \left(\sum_{k\in\mathbb{Z}} \int_{\Omega_k} \left(\left| \frac{b_k}{a_k} f(x)\omega(x)^{\sigma_0} \right| \right)^p d\mu \right)^{1/p} \\
&+ t\left(\sum_{k\in\mathbb{Z}} \int_{\Omega_k} \left(\left| \frac{c_k}{a_k} f(x)\omega(x)^{\sigma_1} \right| \right)^p d\mu \right)^{1/p} \\
&\le \|\{b_k\}\|_{l_p} + t\left(\sum_{k\in\mathbb{Z}} \int_{\Omega_k} \left(\left| \frac{c_k}{a_k} f(x)\omega(x)^{\sigma_0} 2^{(k+1)(\sigma_1-\sigma_0)} \right| \right)^p d\mu \right)^{1/p} \\
&= \|\{b_k\}\|_{l_p} + t2^{\sigma_1-\sigma_0}\|\{2^{k(\sigma_1-\sigma_0)}c_k\}\|_{l_p}
\end{aligned}$$

or

$$K(t,f;L_p^{\sigma_0},L_p^{\sigma_1}) \le 2^{\sigma_1-\sigma_0} K(t,\{a_k\};l_p,l_p^{\sigma_1-\sigma_0}).$$

The equivalence 2.5 and the well known interpolation formula

$$(l_p,l_p^{\sigma_1-\sigma_0}) = l_q^{\theta(\sigma_1-\sigma_0)}$$

(see e. g. [6], Th. 5.6.1) gives

$$\|f\|_{(L_{p_0}^{\sigma_0},L_{p_1}^{\sigma_1})_{\theta,q}} \approx \|\{a_k\}\|_{l_q^{\theta(\sigma_1-\sigma_0)}},$$

and going back from the sequence $\{a_k\}$ to the function f we obtain

$$\begin{aligned}
\|\{a_k\}\|_{l_q^{\theta(\sigma_1-\sigma_0)}} &= \left(\sum_{k\in\mathbb{Z}} (2^{k\theta(\sigma_1-\sigma_0)}a_k)^q \right)^{1/q} \\
&= \left(\sum_{k\in\mathbb{Z}} 2^{kq\theta(\sigma_1-\sigma_0)} \left(\int_{\Omega_k} |f(x)\omega(x)^{\sigma_0}|^p \right)^{q/p} d\mu \right)^{1/q} \\
&\ge 2^{\sigma_1-\sigma_0} \left(\sum_{k\in\mathbb{Z}} \left(\int_{\Omega_k} |\omega(x)^{\theta(\sigma_1-\sigma_0)} f(x)\omega(x)^{\sigma_0}|^p \right)^{q/p} d\mu \right)^{1/q} \\
&= 2^{\sigma_1-\sigma_0}\|f\|_{L_p^{\sigma,q}}
\end{aligned}$$

and, similarly,

$$\|\{a_k\}\|_{l_q^{\theta(\sigma_1-\sigma_0)}} \le \|f\|_{L_p^{\sigma,q}},$$

which means that

$$(L_p^{\sigma_0}, L_p^{\sigma_1})_{\theta,q} = L_p^{\sigma,q}$$

with equivalent norms. The proof is complete. □

Next we prove that if $p_0 \neq p_1$ then the spaces $L_{p,r}^{\sigma,q}$ are stable under diagonal interpolation.

Lemma 2.21. *If $p_0 \neq p_1$, $0 \le q_0,\ q_1 \le \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then*

$$(L_{p_0,r_0}^{\sigma_0,q_0}, L_{p_1,r_1}^{\sigma_1,q_1})_{\theta,q} = L_{p,q}^{\sigma,q},$$

where $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof. Denote the interpolation space $(L_{p_0,r_0}^{\sigma_0,q_0}, L_{p_1,r_1}^{\sigma_1,q_1})_{\theta,q}$ by L and let $g_k(x) = f(x)\omega(x)^{\sigma_0}\chi_{\Omega_k}(x)$.

Then, as $0 \le q_0,\ q_1 \le \infty$ we can use the power theorem (see e.g. [6], Th. 3.11.6) to obtain

$$L^q = ((L_{p_0,r_0}^{\sigma_0,q_0})^{q_0}, (L_{p_1,r_1}^{\sigma_1,q_1})^{q_1})_{\eta,1}, \qquad \text{where } \eta = \theta q/q_1. \tag{2.6}$$

Moreover,

$$(\|f\|_{L_{p_0,r_0}^{\sigma_0,q_0}})^{q_0} = \sum_{k\in\mathbb{Z}} (\|f\omega^{\sigma_0}\chi_{\Omega_k}\|_{L_{p_0,r_0}})_0^q = \sum_{k\in\mathbb{Z}} (\|g_k\|_{L_{p_0,r_0}})^{q_0} \tag{2.7}$$

and according to the mentioned above equivalent representation of $\omega(x)$ on Ω_k,

$$(\|f\|_{L_{p_1,r_1}^{\sigma_1,q_1}})^{q_1} = \sum_{k\in\mathbb{Z}} (\|f\omega^{\sigma_1}\chi_{\Omega_k}\|_{L_{p_1,r_1}})_1^q \tag{2.8}$$

$$= \sum_{k\in\mathbb{Z}} (\|\omega^{\sigma_1-\sigma_0} f\omega^{\sigma_0}\chi_{\Omega_k}\|_{L_{p_1,r_1}})_1^q = \sum_{k\in\mathbb{Z}} 2^{k(\sigma_1-\sigma_0)} (\|g_k\|_{L_{p_1,r_1}})_1^q$$

Note that the family $\{\Omega_k\}$ covers Ω and that Ω_k are not overlapping. Therefore, by using (2.7) and (2.8), we see that

$$K(t, f; (L_{p_0,r_0}^{\sigma_0,q_0})^{q_0}, (L_{p_1,r_1}^{\sigma_1,q_1})^{q_1}) \approx \sum_{k\in\mathbb{Z}} K(t, g_k; (L_{p_0,r_0})^{q_0}, (2^{k(\sigma_1-\sigma_0)} L_{p_1,r_1})^{q_1}).$$

In view of (2.6) and the additivity of the integral it follows that

$$\|f\|_L^q \approx \sum_{k\in\mathbb{Z}} \|g_k\|_{L_k}, \quad \text{where} \quad L_k := ((L_{p_0,r_0})^{q_0}, (2^{k(\sigma_1-\sigma_0)} L_{p_1,r_1})^{q_1})_{\eta,1}. \tag{2.9}$$

By using the power theorem once more we find that

$$((L_{p_0,r_0})^{q_0}, (2^{k(\sigma_1-\sigma_0)} L_{p_1,r_1})^{q_1})_{\eta,1} = ((L_{p_0,r_0}), 2^{k(\sigma_1-\sigma_0)} L_{p_1,r_1})_{\theta,q})^q. \tag{2.10}$$

Next recall that for any quasi-Banach couple,

$$\|x\|_{(X_0,cX_1)_{\theta,q}} = c^{\theta}\|x\|_{(X_0,X_1)_{\theta,q}} \quad \text{for any } c > 0,$$

and therefore by the well known result on interpolation by Lorentz spaces for $p_0 \neq p_1$ (see [6]. Th. 5.3.1) we see that

$$(L_{p_0,r_0}, 2^{k(\sigma_1-\sigma_0)}L_{p_1,r_1})_{\theta,q} = 2^{k\theta(\sigma_1-\sigma_0)}L_{p,q}.$$

Hence, by using (2.4), (2.9), (2.10), we find that

$$\begin{aligned}\|f\|_L^q &= \sum_{k\in\mathbb{Z}}(\|f\omega^{\sigma_0}\chi_{\Omega_k}\|_{2^{k\theta(\sigma_1-\sigma_0)}L_{p,q}})^q \\ &\approx \sum_{k\in\mathbb{Z}}(\|f\omega^{\sigma_0}\omega^{\theta(\sigma_1-\sigma_0)}\chi_{\Omega_k}\|_{L_{p,q}})^q \\ &= \sum_{k\in\mathbb{Z}}(\|f\omega^{\sigma_0}\chi_{\Omega_k}\|_{L_{p,q}})^q = (\|f\|_{L^{\sigma,q}_{p,q}})^q\end{aligned}$$

and the Lemma is proved. □

Remark 2.22. By analyzing the proof of Lemma 3.2 we find that the formula (3.5) is true also for the case $p_0 = p_1$ if we impose the additional condition

$$1/q = (1-\theta)/r_0 + \theta/r_1.$$

In [3] there is an example wich shows that in the off-diagonal case, i.e. when $1/q = (1-\theta)/p_0 + \theta/p_1$ the spaces $(L^{\sigma_0}_{p_0}, L^{\sigma_1}_{p_1})_{\theta,q}$ do not belong in general to the scale of $L^{\sigma,q}_{p.r}$ - spaces, the situation is completely different when one interpolates triples : $(L^{\sigma_0}_{p_0}, L^{\sigma_1}_{p_1}, L^{\sigma_2}_{p_2})$ of weighted L_p spaces, namely, if $0 < q < \infty$, the points $(\sigma_i, 1/p_i)$, $i = 1, 2$ are not colinear, then

$$(L^{\sigma_0}_{p_0}, L^{\sigma_1}_{p_1}, L^{\sigma_2}_{p_2})_{(\theta_1,\theta_2),q} = L^{\sigma,q}_{p,q}$$

where $(\sigma, 1/p) = (1-\theta_1-\theta_2)(\sigma_0, 1/p_0) + \theta_1(\sigma_1, 1/p_1) + \theta_2)(\sigma_2, 1/p_2)$. The proof is based on some lemmas and on so called in the paper first and second reiteration theorems. By using these results, reiteration formulas and results about block-Lorentz spaces, results for interpolation of triples of smoothness index σ_i and integration exponent p_i, $i = 0, 1, 2$ (in particular Besov spaces $B^{\sigma_i}_{p_i}$ and Sobolev spaces $W^{\sigma_i}_{p_i}$ are such) are obtained. The result of real interpolation of such triples is $B^{\sigma,q}_{p,q} = B^{\sigma,q}(L_{p,q})$ - a generalized Besov space based on the Lorentz space. Note, that triples are important here because for couples $(B^{\sigma_0}_{p_0}, B^{\sigma_1}_{p_1})$ of Besov spaces the real interpolation spaces $(B^{\sigma_0}_{p_0}, B^{\sigma_1}_{p_1})_{\theta,q}$ are rather complicated, and except of the diagonal case, they fall outside the scale of Besov spaces, i.e. in general families of smooth function are not in generale stable under real interpolation for couples, but the situation is quite different when we interpolate between triples of smooth functions. Another application of the analogue of the Lions-Peetre reiteration theorem for triples of Banach function lattices is connected with the Stein-Weiss interpolation theorem for L_p spaces with different measures. In 1958 Stein and Weiss proved a very useful interpolation theorem, which can be written as

$$(L_{p_0}(\omega_0 d\mu), L_{p_1}(\omega_1 d\mu))_{\theta,p} = L_p(\omega d\mu)$$

where $p_i \in (0,\infty), \theta \in (0,1),\ 1/p = (1-\theta)/p_0 + \theta/p_1,\ \omega = \omega_0^{(1-\theta)p/p_0}\omega_1^{\theta p/p_1}$. In this connection is natural to ask for an analogue of this formula if we replace L_p spaces with Lorentz $L_{p,q}$ spaces. We note that the nice formula

$(L_{p_0,q_0}(\omega_0 d\mu), L_{p_1,q_1}(\omega_1 d\mu))_{\theta,q} = L_{p,q}(\omega d\mu)$ where $1/q = (1-\theta)/q_0 + \theta/q_1$ is not true in general, it is proved in [3] that the result of such interpolation is a block-Lorentz space. An essential part of the proof uses the second reitereation theorem for triples.

3. On Edmunds - Triebel Spaces

Here we will consider some spaces in which both the interpolation and extrapolation ideas appear. While interpolation theory has serious applications to operator theory (as can be seen at least in monographs by Pietch [62] and Triebel [69]), the connection of extrapolation theory with operator theory is not widely studied. Triebel used in 1993 extrapolation ideas for studying the degree of compactness of some limiting Sobolev embeddings, abstract results (for instance for special operator ideals) have Cobos, Kuhn, Shonbeck, Carro, Fernandez-Cabrera, Martinez, Kryczka.

Let A_0 and A_1 be two complex Banach spaces, such that A_0 is densely continuously embedded in A_1. For every $0 \le \theta \le 1,\ 1 \le q \le \infty$, and $b \in \mathbb{R} \setminus \{0\}$ Edmunds and Triebel have introduced the family of logarithmic spaces $A_\theta(logA)_{b,q}$ in [25]. The Zygmund space $L_p(logL)_b(\Omega)$ is a special case of logarithmic space.

In the first subsection we give some preliminaries about logarithmic spaces. In the second subsection we show how uniform convexity can be preserved in the logarithmic spaces $A_\theta(\log A)_{b,p}$. Estimates are given for the moduli of convexity of $A_\theta(\log A)_{b,p}$ in terms of the moduli of A_0 and A_1, when one or both of them are uniformly convex. In the third one we get estimates of a measure of weak noncompactness of the unit balls of the spaces $A_\theta(logA)_{b,q}$ in terms of the measures of weak noncompactness of the unit balls of the spaces A_0 and A_1. In the fourth subsection we get estimates of the $n-$th Jordan - von Neumann constant C^n_{NJ} and the $n-$th James constant J_n of the spaces $A_\theta(logA)_{b,q}$ in terms of the corresponding constants of the spaces A_0 and A_1. In the fifth subsection we show that the (p,p') Clarkson inequality holds in the Edmunds-Triebel logarithmic spaces $A_\theta(logA)_{b,q}$. As a consequence of these results we also obtain some new information about the types and the cotypes of these spaces. In all these cases we first look how the information about the spaces A_0 and A_1 is inherited by the complex interpolation space $[A_0, A_1]_\theta$ and then we check what this information gives for the extrapolation spaces.

3.1. **About the Construction of Edmunds and Triebel.** In the previous section we considered some pairs (A_0, A_1) such that A_0 and A_1 are Banach spaces embedded in a common topological vector space U and mentioned that the most important among the various constructions of interpolation with respect to a given couple is the complex method leading to the spaces $[A_0, A_1]_\theta$ (where $0 < \theta < 1$) and the real method leading to the spaces $(A_0, A_1)_{\theta,q}$ (where $0 < \theta < 1$ and $0 < q \leq \infty$).

Jawerth and Milman defined in 1991 the Σ_q and Δ_q extrapolation methods for $1 < q < \infty$. According to their definition, a family $(A_i)_{i\in\mathbb{Z}}$ is called strongly compatible if there exist two Banach spaces Δ and Σ such that $\Delta \hookrightarrow A(i) \hookrightarrow \Sigma$ (continuous embeddings) for every $i \in \mathbb{Z}$. The space $\Sigma_q\{A(i)\}_{i\in\mathbb{Z}}$ is the space of all $\alpha \in \Sigma$ for which there exists $(\alpha_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} A(i)$ such that $\sum\limits_{i\in\mathbb{Z}} \|\alpha(i)\|^q_{A(i)} < \infty$ and $\sum\limits_{i\in\mathbb{Z}} \alpha_i$ is (absolutely) convergent to α in Σ. The norm in $\Sigma_q\{A(i)\}_{i\in\mathbb{Z}}$ is defined by $\|\alpha\|_{\Sigma_q\{A(i)\}_{i\in\mathbb{Z}}} = \inf \left(\sum\limits_{i\in\mathbb{Z}} \|\alpha_i\|^q_{A(i)}\right)^{\frac{1}{q}}$, where the infimum is taken over all representations $(\alpha_i)_{i\in\mathbb{Z}}$ of α as above.
The space $\Delta_q\{A(i)\}_{i\in\mathbb{Z}}$ is the space of all $\alpha \in \bigcap_{i\in\mathbb{Z}} A(i)$ with $\sum\limits_{i\in\mathbb{Z}} \|\alpha\|^q_{A(i)} < \infty$.
The norm in $\Delta_q\{A(i)\}_{i\in\mathbb{Z}}$ is defined by $\|\alpha\|_{\Delta_q\{A(i)\}_{i\in\mathbb{Z}}} = \left(\sum\limits_{i\in\mathbb{Z}} \|\alpha\|^q_{A(i)}\right)^{\frac{1}{q}}$.
Let A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, and $[A_0, A_1]_\eta$ be the complex interpolation space for $0 < \eta < 1$. The definition of complex interpolation space is given in the next section.

For every $0 < \theta < 1$, $1 < q < \infty$, and $b \in \mathbb{R} \setminus \{0\}$ the logarithmic space $A_\theta(logA)_{b,q}$ was defined in [25]. These spaces can be regarded as a special case of extrapolation spaces Σ_q and Δ_q as follows:

Definition 3.1. For $b > 0$ the logarithmic space $A_\theta(logA)_{b,q}$ is the space $\Sigma_q\{A(i)\}_{i\in\mathbb{Z}}$, where $A_i = \{0\}$ for $i < J$ and $A(i) = 2^{ib}[A_0, A_1]_{\eta(i)}$ for $i \geq J$, where $J \in \mathbb{N}$ such that $\theta - 2^{-J} > 0$ and $\eta(i) = \theta - 2^{-i}$ for $i \geq J$.
For $b < 0$ the logarithmic space $A_\theta(logA)_{b,q}$ is the space $\Delta_q\{A(i)\}_{i\in\mathbb{Z}}$, where $A(i) = A_1$ with norm $\|a\| = 0$ for $i < J$, and $A(i) = 2^{ib}[A_0, A_1]_{\theta(i)}$ for $i \geq J$, where $J \in \mathbb{N}$ such that $\theta + 2^{-J} < 1$ and $\theta(i) = \theta + 2^{-i}$ for $i \geq J$.

It is clear that different J define isomorphic spaces.

In [25] the following properties of the family of logarithmic spaces were proved.

i) If $0<\theta_0<\theta<\theta_1<1,\ -\infty<b_0<0<b_1<1$ and $1\le q\le\infty$, then

$$A_{\theta_0}\subset A_\theta(logA)_{b_1,q}\subset A_\theta(logA)_{b_0,q}\subset A_{\theta_1}.$$

ii) If $0<\theta<1,\ -\infty<b_0<0<b_1<1$ and $1\le q\le\hat{q}\le\infty$, then

$$A_\theta(logA)_{b_1,q}\subset A_\theta(logA)_{b_1,\hat{q}}\subset A_\theta\subset A_\theta(logA)_{b_0,q}\subset A_\theta(logA)_{b_0,\hat{q}}.$$

As it is noted in [25] the index q is comparatively not so important. Note also that if $0<\theta<1,\ -\infty<b_0<b_1<\infty$ and $1\le q,\ \hat{q}\le\infty$, then

$$A_\theta(logA)_{b_1,q}\subset A_\theta(logA)_{b_0,\hat{q}}.$$

Many classical spaces are isomorphic to logarithmic spaces. For instance, if Ω is a bounded open subset of R^n with Lebesque n-measure $\mu(\Omega)<\infty,\ 1\le p<\infty$ and $b\in\mathbb{R}$, then the usual Zygmund space $L_p(LogL)_b(\Omega)$ (i.e. the set of all measurable functions $f:\Omega\to C$ such that $\int_\Omega|f(x)|^p\log^{bp}(2+|f(x)|)dx<\infty$) is isomorphic to the logarithmic space $A_\theta(logA)_{b,p}$, where $A_0=L_\infty(\Omega),\ A_1=L_1(\Omega)$ and $\theta=p^{-1}$(see [25]). This space was used in certain limiting situations in spectral theory in [25]. Sometimes it is more convenient (for instance if $1\le p\le 2$) to take $A_0=L_2(\Omega),\ A_1=L_1\Omega),\ \theta=\frac{2-p}{p}$ and a slightly modified variant of $A_\theta(logA)_{b,p}$ to get $L_p(LogL)_b(\Omega)$. In [25] also the related logarithmic Sobolev spaces $H^s_p(LogH)_b(\Omega)$ are considered.

3.2. **Estimates of the modulus of convexity for Edmunds-Triebel spaces.** Let X be a normed space with $\dim X\ge 2$ and $B_X=\{x\in X:\|x\|\le 1\}$ be the unit ball of X.

The modulus of convexity $\delta_X(\varepsilon)$ of X, for $0\le\varepsilon\le 2$, is defined by

$$\delta_X(\varepsilon)=\inf\{1-\frac{\|x+y\|}{2}:x,y\in B_X,\|x-y\|\ge\varepsilon\}.$$

The modulus of smoothness $\rho_X(\tau)$ of X, for $\tau>0$, is defined by

$$\rho_X(\tau)=\sup\{\frac{\|x+y\|+\|x-y\|}{2}-1:x\in B_X,\|y\|\le\tau\}.$$

The space X is said to be uniformly convex (u.c.) if $\delta_X(\varepsilon)>0$ for every $\varepsilon>0$ and uniformly smooth (u.s.)if $\lim\limits_{\tau\to 0}\frac{\rho_X(\tau)}{\tau}=0$.

It is known that $\rho_{X^*}(\tau)=\sup\{\frac{\tau\epsilon}{2}-\delta_X(\varepsilon):0\le\varepsilon\le 2\}$, for every $\tau>0$, where X^* is the dual space of X. We say that a u.c. (u.s.) space X has modulus of convexity (resp. smoothness) of power type p if there exists c, $0<c<+\infty$, such that $\delta_X(\varepsilon)\ge c\varepsilon^p$, (resp. $\rho_X(\tau)\le c\tau^p$). The modulus of convexity of X is of power

type p if and only if the modulus of smoothness of X^* is of power type q, where $\frac{1}{p}+\frac{1}{q}=1$.
For $\varepsilon \geq 0$, the quantity

$$\widetilde{\delta}_X(\varepsilon)=\sup\{\frac{\tau\epsilon}{2}-\rho_X^*(\tau):\tau\geq 0\}$$

was defined in [27] and it was proved that the function $\widetilde{\delta}_X$ is the maximal convex function minorizing δ_X; also, $\widetilde{\delta}_X$ satisfies $\widetilde{\delta}_X(\varepsilon)\geq(\gamma^{-1}-1)\delta(\gamma\varepsilon)$ for every $0<\gamma<1$, and $\varepsilon\geq 0$.

Lemma 3.2. *Let (X_n) be a sequence of Banach spaces and $1<p<+\infty$. For every ε, $0\leq\varepsilon\leq 2$, we put $\delta'(\varepsilon)=\inf\limits_n \inf\limits_{(\frac{\varepsilon}{2})^p\leq s\leq 1} s\delta_{X_n}(\frac{\varepsilon}{s^{\frac{1}{p}}})$ and $\widetilde{\delta'}(\varepsilon)=\sup\limits_{\tau\geq 0}\{\frac{1}{2}\tau\varepsilon-\rho'(\tau)\}$, where $\rho'(\tau)=\sup\limits_n\sup\limits_{u\leq 1} u^q\rho_{X_n^*}(\frac{\tau}{u})$, $\tau\geq 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\widetilde{\delta'}(\varepsilon)\geq(\gamma^{-1}-1)\delta'(\gamma\varepsilon),$$

whenever $0<\gamma<1$ and $0<\varepsilon\leq 2$.

Proof. It is obvious that

$$\sup_{u\leq 1}u^q\rho_{X_n^*}(\frac{\tau}{u})=\sup_{u\leq 1}u^q\sup_{0\leq\varepsilon\leq 2}\{\frac{1}{2}\frac{\tau}{u}\varepsilon-\delta_{X_n}(\varepsilon)\}=\sup_{0\leq\varepsilon\leq 2}\sup_{u\leq 1}\{\frac{1}{2}\tau u^{q-1}\varepsilon-u^q\delta_{X_n}(\varepsilon)\}$$

$$=\sup_{0\leq\varepsilon\leq 2}\sup_{(\frac{\varepsilon}{2})^{\frac{1}{q-1}}\leq u\leq 1}\{\frac{1}{2}\tau\varepsilon-u^q\delta_{X_n}(\frac{\varepsilon}{u^{q-1}})\}=\sup_{0\leq\varepsilon\leq 2}\sup_{(\frac{\varepsilon}{2})^p\leq s\leq 1}\{\frac{1}{2}\tau\varepsilon-s\delta_{X_n}(\frac{\varepsilon}{s^{\frac{1}{p}}})\}$$

$$=\sup_{0\leq\varepsilon\leq 2}\{\frac{1}{2}\tau\varepsilon-\inf_{(\frac{\varepsilon}{2})^p\leq s\leq 1}s\delta_{X_n}(\frac{\varepsilon}{s^{\frac{1}{p}}})\}.$$

From the above we obtain

$$\rho'(\tau)=\sup_n\sup_{0\leq\varepsilon\leq 2}\{\frac{1}{2}\tau\varepsilon-\inf_{(\frac{\varepsilon}{2})^p\leq s\leq 1}s\delta_{X_n}(\frac{\varepsilon}{s^{\frac{1}{p}}})\}\sup_{0\leq\varepsilon\leq 2}\{\frac{1}{2}\tau\varepsilon-\delta'(\varepsilon)\}.$$

So $\widetilde{\delta'}(\varepsilon)\leq\delta'(\varepsilon)$ for every $0\leq\varepsilon\leq 2$. Since $\frac{\delta(\varepsilon)}{\varepsilon}$ is not decreasing ([42]) we obtain that $\frac{\delta'(\varepsilon)}{\varepsilon}$ is nondecreasing. It it easy to see that $\widetilde{\delta'}$ is the maximal convex function minorizing δ'. So, from lemma 2 of [27], we have the conclusion.

Proposition 3.3. *Let $1<p<+\infty$. There exists $K>0$, which depends on p, such that for every sequence of Banach spaces (X_n), if $X=[\sum\limits_{n=1}^{\infty}X_n]_p$ then we have*

$$\delta_X(\varepsilon)\geq 2^{-p}K^{1-p}\varepsilon^p\inf_{\frac{\varepsilon}{2K}\leq t\leq 2}\frac{\delta(t)}{t^p}$$

where $\delta(t)=\inf\limits_n\delta_{X_n}(t)$.

Proof. Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of Banach spaces. If $\tau \geq 0$ and $0 \leq \varepsilon \leq 2$ we put $\rho'(\tau) = \sup_n \sup_{u\leq 1} u^q \rho_{X_n^*}(\frac{\tau}{u})$, $\delta'(\varepsilon) = \inf_n \inf_{(\frac{\varepsilon}{2})^p \leq s \leq 1} s\delta_{X_n}(\frac{\varepsilon}{s^{\frac{1}{p}}})$ and $\widetilde{\delta'}(\varepsilon) = \sup_{\tau\geq 0}\{\frac{1}{2}\tau\varepsilon - \rho'(\tau)\}$, where $\frac{1}{p}+\frac{1}{q} = 1$. From Proposition 19 of [27] there exists $K > 0$, which depends only on p, such that $\rho_{X^*}(\varepsilon) \leq K\rho'(\tau)$. So, from the previous lemma we obtain

$$\delta_X(\varepsilon) \geq \widetilde{\delta}_X(\varepsilon) \geq K \sup_{\tau\geq 0}\{\frac{1}{2}\tau\varepsilon K^{-1} - \rho'(\tau)\} = K\widetilde{\delta'}(\frac{\varepsilon}{K}) \geq K\delta'(\frac{\varepsilon}{2K})$$

$$= K \inf_n \inf_{(\frac{\varepsilon}{4K})^p \leq s \leq 1} s\delta_{X_n}(\frac{\varepsilon}{2Ks^{\frac{1}{p}}}) = K \inf_n \inf_{(\frac{\varepsilon}{4K})^p \leq s \leq 1} \frac{\delta_{X_n}(\frac{\varepsilon}{2Ks^{\frac{1}{p}}})}{\frac{1}{s}}$$

$$= K \inf_n \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} (\frac{\varepsilon}{2K})^p \frac{\delta_{X_n}(t)}{t^p} = 2^{-p}K^{1-p}\varepsilon^p \inf_{\frac{\varepsilon}{2K}\leq t\leq 2} \frac{\delta(t)}{t^p}.$$

Remarks. i) For each $1 \leq p \leq 2$ and $0 < \varepsilon \leq \eta$, it is easy to see that $\frac{\delta(\varepsilon)}{\varepsilon^p} \leq 4L\frac{\delta(\eta)}{\eta^p}$, where L is a constant less than 3.18 (see [27], Corollary 11). So if $1 < p \leq 2$ and $(X_n), X$ and $\delta(\varepsilon)$ are as in Proposition 3.3, then we obtain $\delta_X(\varepsilon) \geq 8KL\delta(\frac{\varepsilon}{2K})$, where K is a constant which depends only on p and L is the constant of Corollary 11 of [27].
ii) Whenever $1 < p, k < +\infty$ we conclude from Proposition 3.3 that, if the spaces (X_n) have modulus of convexity of power type k uniformly, that is there is $0 < c < +\infty$ such that $\delta(X_n) \geq c\varepsilon^k$ for every $n \in \mathbf{N}$, then the space $[\sum_{n=1}^{\infty} X_n]_p$ has modulus of convexity of power type $\max\{p, k\}$.
iii)Let Ω be a bounded open subset of $\mathbf{R}^n$, $1 \leq p < +\infty$ and $b \in \mathbf{R}\setminus\{0\}$. An infinite family of equivalent norms was defined on the Zygmund spaces $L_p(\log L)_b(\Omega)$ in [25]. Using ii) it is easy to see the following:
For $b < 0$ and $1 < p \leq 2$ (resp. $2 < p$), the Zygmund spaces have modulus of convexity of power type 2 (resp. p) for all of these norms.
For $b > 0$, if $p < 2$ then the Zygmund spaces have modulus of convexity of power type 2 for each of these norms; if $2 \leq p$ then they have modulus of convexity of power type r for every $r > p$, for an infinite number of these norms.

Given an interpolation couple (A_0, A_1), if at least one of A_0, A_1 is u.c. then the space A_θ is u.c. for every $0 < \theta < 1$ [22]. The converse does not hold. However if one of the interpolation spaces, let say A_θ is u.c. then all of them are. Let us prove this.

Let i) $\theta < \theta' < 1$. We put $\eta = \frac{\theta'-\theta}{1-\theta}$. Since $A_1 = [A_0, A_1]_1$, using the reiteration theorem (see [6]), we find that $[A_0, A_1]_{\theta'} = [A_\theta, A_1]_\eta$. From [22], since A_θ is u.c.

we have

$$\delta_{A_{\theta'}}(\varepsilon) \geq (1-\eta)\delta_{A_\theta}(2(\frac{\varepsilon}{2})^{\frac{1}{1-\eta}}) = \frac{1-\theta'}{1-\theta}\delta_{A_\theta}(2(\frac{\varepsilon}{2})^{\frac{1-\theta}{1-\theta'}}).$$

Let ii) if $0 < \theta' < \theta$ then

$$\delta_{A_{\theta'}}(\varepsilon) \geq \frac{\theta'}{\theta}\delta_{A_\theta}(2(\frac{\varepsilon}{2})^{\frac{\theta}{\theta'}}).$$

Let $J \in \mathbf{N}, j \geq J$ and $\theta + 2^{-J} < 1$ and $\theta_j = \theta + 2^{-j}$. Applying the above estimates for $\theta' = \theta_j$ and since $\delta_{A_{\theta_j}}$ is nondecreasing, we obtain

$$\delta_{A_{\theta_j}}(\varepsilon) \geq \frac{1-\theta-2^{-J}}{1-\theta}\delta_{A_\theta}\left(2(\frac{\varepsilon}{2})^{\frac{1-\theta}{1-\theta-2^{-J}}}\right) \tag{3.1}$$

for every $j \geq J$. Analoguously, if $\theta - 2^{-J} > 0$ and $\eta_j = \theta - 2^{-j}$ for every $j \geq J$, then

$$\delta_{A_{\eta_j}}(\varepsilon) \geq \frac{\theta-2^{-J}}{\theta}\delta_{A_\theta}\left(2(\frac{\varepsilon}{2})^{\frac{\theta}{\theta-2^{-J}}}\right) \tag{3.2}$$

For every $J \in \mathbf{N}$ and $b \in \mathbf{R}\backslash\{0\}$ we put $c(J,b) = \frac{1-\theta-2^{-J}}{1-\theta}$ if $b < 0$ and $c(J,b) = \frac{\theta-2^{-J}}{\theta}$ if $b > 0$.

Theorem 3.4. *Let $1 < p < +\infty$, $A_0 \subset A_1$ be two complex Banach spaces, $0 < \theta < 1$ and $b \in \mathbf{R}\backslash\{0\}$. If the space A_θ is u.c. then the logarithmic space $A = A_\theta(\log A)_{b,p}$ is u.c. In particular, there exists $K > 0$, which depends only on p, such that for every $J \in \mathbf{N}$, if we consider the space A with the J-norm, we have*

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}c(J,b)\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\delta_{A_\theta}\left(2(\frac{t}{2})^{\frac{1}{c(J,b)}}\right)}{t^p}.$$

Proof. Let $J \in \mathbf{N}$.

i) Let $b < 0$. Then (3.1) can be written like

$$c(J,b)\delta_{A_\theta}(2(\frac{\varepsilon}{2})^{\frac{1}{c(J,b)}}) \leq \delta_{A_{\theta_j}}(\varepsilon)$$

for every $j \geq J$. From the above inequality, the definition of the logarithmic space and Proposition 3.3 we obtain

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\inf\limits_{J \leq j < +\infty} \delta_{A_{\theta_j}}(t)}{t^p} \geq 2^{-p}K^{1-p}c(J,b)\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\delta_{A_\theta}\left(2(\frac{t}{2})^{\frac{1}{c(J,b)}}\right)}{t^p}.$$

So we have the conclusion.

ii) Let $b > 0$ and $\alpha, \beta \in A$ be such that $\| \alpha \|, \| \beta \| < 1$ and $\| \alpha - \beta \| > \varepsilon$. Then, for every $j \geq J$ there exist $\alpha_j, \beta_j \in A_{\eta_j}$, such that $\alpha = \sum\limits_{j=J}^{+\infty} \alpha_j$, $\beta = \sum\limits_{j=J}^{+\infty} \beta_j$ and

$\sum_{j=J}^{+\infty} 2^{jbp} \parallel \alpha_j \parallel^p < 1$, $\sum_{j=J}^{+\infty} 2^{jbp} \parallel \beta_j \parallel^p < 1$. Then we have $\left(\sum_{j=J}^{+\infty} 2^{jbp} \parallel \alpha_j - \beta_j \parallel^p\right)^{\frac{1}{p}} >$ ε. We put $X = [\sum_{j=J}^{+\infty} \bigoplus A_{\eta_j}]_p$, where we consider the space A_{η_j} with the norm $2^{jbp}\|.\|_{A_{\eta_j}}$. Then, setting $x = (\alpha_j)$, $y = (\beta_j)$, we have $x,\ y \in X$, $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \varepsilon$. So

$$\|x + y\| \leq 2(1 - \delta_X(\varepsilon)).$$

From this we have $\|\alpha+\beta\| \leq 2(1-\delta_X(\varepsilon))$ and so, from Proposition 3.3, we obtain

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\inf_{J \leq j < +\infty} \delta_{A_{\eta_j}}(t)}{t^p}.$$

From the inequality (3.2) it follows that

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}c(J,b)\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\delta_{A_\theta}\left(2(\frac{t}{2})^{\frac{1}{c(J,b)}}\right)}{t^p}.$$

Corollary 3.5. *Let $A_0 \subset A_1$ be two complex Banach spaces. If there exists θ_0, $0 < \theta_0 < 1$, such that the complex interpolation space A_{θ_0} is u.c., then the logarithmic space $A_\theta(\log A)_{b,p}$ is u.c. for every $0 < \theta < 1$, $1 < p < \infty$ and $b \in \mathbf{R}\backslash\{0\}$.*

Proof. From the inequalities (3.1) and (3.2) we obtain that $[A_0, A_1]$ is u.c. So, the result follows from Theorem 3.4.

The next corollary follows from Theorem 3.4 and [22].

Corollary 3.6. *Let A_0, A_1, p, b, A_θ and A as in Theorem 3.4. Then there exists $K > 0$, which depends only on p, such that for every $J \in \mathbf{N}$, if we consider the space A with the J-norm, we have:*
i) If the space A_0 is u.c. then

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}(1-\theta)c(J,b)\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\delta_{A_0}\left(2(\frac{t}{2})^{\frac{1}{c(J,b)(1-\theta)}}\right)}{t^p}.$$

ii) If the space A_1 is u.c. then

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}\theta c(J,b)\varepsilon^p \inf_{\frac{\varepsilon}{2K} \leq t \leq 2} \frac{\delta_{A_1}\left(2(\frac{t}{2})^{\frac{1}{c(J,b)\theta}}\right)}{t^p}.$$

iii) If the spaces A_0 and A_1 are u.c. then

$$\delta_A(\varepsilon) \geq 2^{-p}K^{1-p}c(J,b)\varepsilon^p \inf_{\frac{\varepsilon}{2K}\leq t\leq 2} \frac{\delta\left(min\{\theta, 1-\theta\}(\frac{t}{2})^{\frac{1}{c(J,b)}}\right)}{t^p},$$

where $\delta(\varepsilon) = [(\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}]^{-1}(\varepsilon)$ and $\phi_i(\varepsilon) = \frac{1}{26}\delta_{A_i}(\frac{\varepsilon}{2\sqrt{2}})$ for $i = 0, 1$.

Proof. i) This follows from Theorem 3.4 and [22] (Theorem 1(i)), since $\delta_{A_{\theta(\varepsilon)}} \geq (1-\theta)\delta_{A_0}(2(\frac{\varepsilon}{2})^{\frac{1}{1-\theta}})$.
ii) This follows from Theorem 3.4and [22] (Theorem 1(i)), since $\delta_{A_{\theta(\varepsilon)}} \geq \theta\delta_{A_1}(2(\frac{\varepsilon}{2})^{\frac{1}{\theta}})$.
iii) The constants of Theorem 1.e.9 of [42] can be estimated using [27] (Corollary 11). In particular, we obtain that $\delta_{L_2(X)} \geq \frac{1}{26}\delta_X(\frac{\varepsilon}{2\sqrt{2}})$. So, from the proof of Theorem 1(ii) of [22], we have that $\delta_{A_\theta}(\varepsilon) \geq \delta(\frac{min\{\theta,1-\theta\}}{2}\varepsilon)$, where $\delta(\varepsilon) = [(\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}]^{-1}(\varepsilon)$ and $\phi_i(\varepsilon) = \frac{1}{26}\delta_{A_i}(\frac{\varepsilon}{2\sqrt{2}})$ for $i = 0, 1$. The result follows from the above and Theorem 3.4.

3.3. **On the measure of weak noncompactnes of Edmunds-Triebel spaces.** Let X be a Banach space and M_X the family of all nonempty bounded subsets of X. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $u_1, u_2 \in X$, then u_1 and u_2 are said to be a pair of successive convex combinations (scc) for $(x_n)_{n\in\mathbb{N}}$ if $u_1 \in conv\{x_1, ..., x_r\}$ and $u_2 \in conv\{x_{r+1}, x_{r+2}, ...\}$ for some integer $r \geq 1$. For every $M \in M_X$ the measure of weak noncompactness $\gamma(M)$ defined in [39] is given by

$$\gamma(M) = \sup\{csep(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in convM\},$$

where $csep(x_n)_{n\in\mathbb{N}} = \inf\{\|u_1 - u_2\| : u_1,\ u_2 \text{ are scc for } (x_n)_{n\in\mathbb{N}}\}$.
The measure of weak noncompactness γ is related to the well-known James criterion:
A weakly closed $M \subset X$ is not weakly compact iff there exists $\delta > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in M, such that $dist(conv\{x_1, ..., x_r\}, conv\{x_{r+1}, x_{r+2}, ...\}) \geq \delta$ for every $r \in \mathbb{N}$.
From this criterion it is clear that $\gamma(M) = 0$ iff M is relatively weakly compact.
The measure γ coincides with the function measuring the deviation from relative weak compactness based on the double-limit criterion, considered in [5]. Namely,

$$\gamma(M) = \sup\{\lim_{m\to\infty}\lim_{n\to\infty} f_m(x_n) - \lim_{n\to\infty}\lim_{m\to\infty} f_m(x_n) : (x_n)_{n\in\mathbb{N}} \subset M,\ (f_m)_{m\in\mathbb{N}} \subset B_{X^*} \text{ and the limits exist}\}.$$

So, $\gamma(M)$ is the worst distance between iterated limits for sequences in M and sequences in the dual unit ball B_{X^*}.

Another measure of weak noncompactness was introduced by de Blasi [8]. This measure is given by the formula

$$\omega(M) = \inf\{t > 0 : M \subset C + t\overline{B_X}, C \subset X \text{ is weakly compact}\}$$

for each $M \in M_X$. Hence, $\omega(M)$ is the worst distance from M to weakly compact sets of X. This measure was succesfully applied to operator theory and to the theory of differential and integral equations. Logarithmically convex estimates for the measure of weak noncompactness ω have been established by Askoj and Maligranda [4], Cobos and Martinez [17]. In [1] relations between the measures of weak noncompactness γ and ω were proved.

While the γ is a counterpart of separation measure of non compactness, de Blasi measure appears as a counterpart for the weak topology of Hausdorff measure of noncompactness. We have $\gamma(M) \leq 2\omega(M)$ in general, but γ is not equivalent to ω (see [1, 5, 38]). They coincide in c_0([39]), and if M is a nonempty bounded subset of $L_1(\mu)$, where μ is a finite measure, then $\gamma(M) = 2\omega(M)$.

For every bounded operator $T : E \to F$ the number $\Gamma(T) = \gamma(T(B_E))$ is called measure of weak noncompactness of the operator T. For weak topologies Gantmacher established that the operator $T : E \to F$ is weakly compact iff T^* is weakly compact. The quantitative result is $\gamma(T(B_E)) \leq \gamma(T^*(B_{F^*})) \leq 2\gamma(T(B_E))$ [1]. From [5, theorem 4] we obtain that there are no constants m and M, such that $m\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq M\omega(T(B_E))$ for any bounded operator $T : E \to F$.

For more details about the measure of weak noncompactness γ see [1, 38, 39].

The following result was proved in [38].

Theorem 3.7. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two Banach couples, $0 < \theta < 1$ and $T : \overline{A} \to \overline{B}$. Then*

$$\Gamma_{[\theta]}(T) \leq \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^{\theta},$$

where $\Gamma_{[\theta]}(T)$ and $\Gamma_j(T), j = 0, 1,$ are the measures of weak noncompactness Γ of the operators $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ and $T : A_j \to B_j,\ j = 0, 1,$ respectively.

If $\{A(i)\}_{i\in\mathbb{Z}}$ and $\|(B_i)\}_{i\in\mathbb{Z}}$ are two strongly compatible families, $T : \{A(i)\}_{i\in\mathbb{Z}} \to \{B(i)\}_{i\in\mathbb{Z}}$ and $1 < q < \infty$, then we denote by $\Gamma_{\Sigma_q}(T)$ (resp. $\Gamma_{\Delta_q}(T)$) the measures of weak noncompactness Γ of the operator $T : \Sigma_q(A(i)_{i\in\mathbb{Z}}) \to \Sigma_q(B(i)_{i\in\mathbb{Z}})$ (resp. $T : \Delta_q(A(i)_{i\in\mathbb{Z}}) \to \Delta_q(B(i)_{i\in\mathbb{Z}})$). Then, we can write Theorem 4.1 in [37] as follows.

Theorem 3.8. *Let $\{A(i)\}_{i\in\mathbb{Z}}$ and $\{B(i))\}_{i\in\mathbb{Z}}$ be two strongly compatible families, $T : \{A(i)\}_{i\in\mathbb{Z}} \to \{B(i)\}_{i\in\mathbb{Z}}$ and $1 < q < \infty$. Then*

i) $\Gamma_{\Sigma_q}(T) \leq \sup\{\Gamma(T : A_i \to B_i) : i \in \mathbb{Z}\}$, *and*
ii) $\Gamma_{\Delta_q}(T) \leq \sup\{\Gamma(T : A_i \to B_i) : i \in \mathbb{Z}\}$.

Using Theorems 3.7 and 3.8 we can prove the following result concerning an estimate of the measure of weak noncompactness of an operator between logarithmic spaces.

Theorem 3.9. *Let A_0, A_1, B_0, B_1 be Banach spaces such that A_0 is densely and continuously embedded into A_1 and B_0 is densely and continuously embedded into B_1, $0 < \theta < 1$, $1 < q < \infty$ and $b \in \mathbb{R} \setminus \{0\}$. Let also $T : A_1 \to B_1$ be a bounded operator such that $T(A_0) \subseteq B_0$ and $\Gamma_j(T)$, $j = 0, 1$, be the measures of weak noncompactness Γ of the operators $T : A_j \to B_j$, $j = 0, 1$.*
i) If $\Gamma_0(T) = 0$, or $\Gamma_1(T) = 0$, then $\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) = 0$.
ii)If $\Gamma_0(T)\Gamma_1(T) \neq 0$, then
a) for $b < 0$

$$\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) \leq \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta \max\left(1, \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{2^{-J}}\right),$$

and
b) for $b > 0$

$$\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) \leq \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta \max\left(1, \left(\frac{\Gamma_0(T)}{\Gamma_1(T)}\right)^{2^{-J}}\right),$$

where J is the integer from the definitions of $A_\theta(logA)_{b,q}$.

Proof. i) If $\Gamma_0(T) = 0$, or $\Gamma_1(T) = 0$, from Theorems 3.7 and 3.8 we obtain that $\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) = 0$.
ii) Let $\Gamma_0(T)\Gamma_1(T) \neq 0$, $b < 0$, $J \in \mathbb{N}$, such that $\theta + 2^{-J} < 1$, and $\theta(i) = \theta + 2^{-i}$ for $i \geq J$. We put $A(i) = [A_0, A_1]_{\theta(i)}$, $B(i) = [B_0, B_1]_{\theta(i)}$ and $C = \sup_{i\in\mathbb{Z}}\{\Gamma(T : A(i) \to B(i))\}$. By Theorems 3.7 and 3.8 we get $C \leq \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{\theta(J)} \Gamma_0(T)$. If $\frac{\Gamma_1(T)}{\Gamma_0(T)} \leq 1$, then $C \leq \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{\theta} \Gamma_0(T) = \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta$. If $\frac{\Gamma_1(T)}{\Gamma_0(T)} \geq 1$, then $C \leq \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{\theta+2^{-J}} \Gamma_0(T) = \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{2^{-J}}$.
So, the result follows from Theorem 3.8.
The proof for the case $b > 0$ is analogue.

From Theorem 3.9 we obtain the following Corollaries.

Corollary 3.10. *Let A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, $0 < \theta < 1$, $1 < q < \infty$ and $b \in \mathbb{R} \setminus \{0\}$.*
i)$\gamma(B_{A_\theta(logA)_{b,q}}) \leq \gamma(B_{A_0})^{(1-\theta)}\gamma(B_{A_1})^\theta$
ii) If A_0 or A_1 is reflexive, then the space $A_\theta(logA)_{b,q}$ is also reflexive.

Corollary 3.11. *Let* $A_0,\ A_1,\ B_0,\ B_1,\ T,\ \theta, b$ *and* q *be as Theorem 3.9. If one of the operators* $T : A_j \to B_j,\ j = 0, 1,$ *is weakly compact, then the operator* $T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}$ *is also weakly compact.*

Let us mention that as far as we know in a submitted paper Fernandez-Cabrera and Martinez improved this corollaries.

3.4. **On n-th James and Jordan-von Neumann constants for Edmunds-Triebel spaces.** In order to estimate the n-th Jordan - von Neumann constant and James constant of a logarithmic space we prove two Lemmas concerning estimations of the n-th Jordan - von Neumann constants of interpolation and extrapolation spaces.

Lemma 3.12. *If* (A_0, A_1) *is couple of Banach spaces and* $0 < \theta < 1$*, then* $C^{(n)}_{NJ}([A_0, A_1]_\theta) \le C^{(n)}_{NJ}(A_0)^{1-\theta} C^{(n)}_{NJ}(A_1)^\theta$.

Proof. Let $T : [\ell^2_n(A_0) \bigoplus \ell^2_n(A_1)]_1 \to [\ell^2_{2^n}(A_0) \bigoplus \ell^2_{2^n}(A_1)]_1$ be defined by
$T\left((x_1, ..., x_n), (y_1, ..., y_n)\right) = \left(\left(\sum_{i=1}^n \theta_i x_i\right)_{\theta_i \in \{-1,1\}}, \left(\sum_{i=1}^n \theta_i y_i\right)_{\theta_i \in \{-1,1\}} \right)$.
Then $T(\ell^2_n(A_i)) \subseteq \ell^2_{2^n}(A_i)$ and $\|T|\ell^2_n(A_i)\| = \sqrt{2^n C^{(n)}_{NJ}(A_i)}$ for $i = 0, 1$.
Since $[\ell^2_n(A_0), \ell^2_n(A_1)]_\theta = \ell^2_n([A_0, A_1]_\theta)$ and $[\ell^2_{2^n}(A_0), \ell^2_{2^n}(A_1)]_\theta = \ell^2_{2^n}([A_0, A_1]_\theta)$ we obtain that $T(\ell^2_n([A_0, A_1]_\theta)) \subseteq \ell^2_{2^n}([A_0, A_1]_\theta)$ and
$\|T|\ell^2_n([A_0, A_1]_\theta)\| \le \sqrt{2^n C^{(n)}_{NJ}(A_0)^{1-\theta} C^{(n)}_{NJ}(A_1)^\theta}$. But $\|T|\ell^2_n([A_0, A_1]_\theta)\|$
$= \sqrt{2^n C^{(n)}_{NJ}([A_0, A_1]_\theta)}$. Thus $C^{(n)}_{NJ}([A_0, A_1]_\theta) \le C^{(n)}_{NJ}(A_0)^{1-\theta} C^{(n)}_{NJ}(A_1)^\theta$. □

Next we prove that $C^{(n)}_{NJ}(\Sigma_q((A_i)_{i\in\mathbb{Z}}))$ and $C^{(n)}_{NJ}(\Delta_q((A_i)_{i\in\mathbb{Z}}))$ do not exceed $n^{2/t-1} \sup\limits_{i\in\mathbb{Z}} C^{(n)}_{NJ}(A_i)^{2/t'}$, where $t = \min\{q, q'\}, 1 < q < \infty$.

Indeed put $C_i = C^{(n)}_{NJ}(A_i)$ for $i \in \mathbb{Z}$ and $C = \sup\limits_{i\in\mathbb{Z}} C_i$. From [33] we obtain $C^{(n)}_{NJ}(\ell_q((A_i))) \le n^{2/t-1} C^{2/t'}$.
Let $n \ge 2,\ \alpha_1, ..., \alpha_n \in \Sigma_q((A_i))$ with $\sum_{j=1}^n \|\alpha_j\| \ne 0$, and $\varepsilon > 0$. For every $j = 1, ..., n$ there exists a representation $(\alpha_j(i))_{i\in\mathbb{Z}}$ of α_j such that $\left(\sum_{i\in\mathbb{Z}} \|\alpha_j(i)\|^q_{A_i}\right)^{\frac{1}{q}} - \|\alpha_j\|_{\Sigma_q} < \varepsilon$. Then

$$\sum_{\theta_j\in\{-1,1\}} \left\| \sum_{j=1}^n \theta_j \alpha_j \right\|^2_{\Sigma_q} \le \sum_{\theta_j\in\{-1,1\}} \left(\sum_{i\in\mathbb{Z}} \left\| \sum_{j=1}^n \theta_j \alpha_j(i) \right\|^q_{A_i} \right)^{\frac{2}{q}} = \sum_{\theta_j\in\{-1,1\}} \left\| \sum_{j=1}^n \theta_j \alpha_j \right\|^2_{\ell_q(A_i)} \le$$

$$\le 2^n n^{2/t-1} C^{2/t'} \sum_{j=1}^n \|\alpha_j\|^2_{l_q(A_i)} \le 2^n n^{2/t-1} C^{2/t'} \sum_{j=1}^n (\|\alpha_j\|_{\Sigma_q} + \varepsilon)^2.$$

So, we get

$$\sum_{\theta_j\in\{-1,1\}} \left\| \sum_{j=1}^{n} \theta_j \alpha_j \right\|_{\Sigma_q}^2 \leq 2^n n^{2/t-1} C^{2/t'} \sum_{j=1}^{n} \|\alpha_j\|_{\Sigma_q}^2.$$

This gives the inequality

$$C_{NJ}^{(n)}(\Sigma_q((A_i)_{i\in\mathbb{Z}})) \leq n^{2/t-1} \sup_{i\in\mathbb{Z}} C_{NJ}^{(n)}(A_i)^{2/t'}.$$

The proof of the inequality $C_{NJ}^{(n)}(\Delta_q((A_i)_{i\in\mathbb{Z}})) \leq n^{2/t-1} \sup\limits_{i\in\mathbb{Z}} C_{NJ}^{(n)}(A_i)^{2/t'}$ is similar.

Then acting in a way similar to the proof of Theorem 3.9 we come to

Theorem 3.13. *Let A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, $0<\theta<1$, $b\in\mathbb{R}\setminus\{0\}$, $1<q<\infty$, $t=\min\{q,q'\}$ and J be the integer from the definition of the logarithmic space $A=A_\theta(logA)_{b,q}$. i)If $b<0$, then*

$$C_{NJ}^{(n)}(A) \leq n^{2/t-1} C_{NJ}^{(n)}(A_0)^{\frac{2(1-\theta)}{t'}} C_{NJ}^{(n)}(A_1)^{\frac{2\theta}{t'}} \max\left\{1, \left(\frac{C_{NJ}^{(n)}(A_1)}{C_{NJ}^{(n)}(A_0)}\right)^{\frac{2^{-J+1}}{t'}}\right\}$$

ii) If $b>0$, then

$$C_{NJ}^{(n)}(A) \leq n^{2/t-1} C_{NJ}^{(n)}(A_0)^{\frac{2(1-\theta)}{t'}} C_{NJ}^{(n)}(A_1)^{\frac{2\theta}{t'}} \max\left\{1, \left(\frac{C_{NJ}^{(n)}(A_0)}{C_{NJ}^{(n)}(A_1)}\right)^{\frac{2^{-J+1}}{t'}}\right\}.$$

About the James constant, we first prove

Theorem 3.14. *Let (A_0,A_1) be a couple of Banach spaces, $0<\theta<1$ and $A_\theta=[A_0,A_1]_\theta$. Then*

$$\frac{J_n(A_\theta)}{n} \leq \left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^n}} \left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^n}}$$

Proof. We put $\beta_0=J_n(A_0)$, $\beta_1=J_n(A_1)$. Let $0<q<1$. We will prove that for any $x_1,\dots,x_n\in B_{A_\theta}$ there exist $\varepsilon_1,\dots,\varepsilon_n=\pm1$ such that

$$\|\sum_{k=1}^{n}\varepsilon_k x_k\| \leq B_n = \beta_0^{\frac{(1-\theta)q}{2^n}} \beta_1^{\frac{\theta q}{2^n}} n^{1-\frac{q}{2^n}}.$$

Then considering $q\to1$, we will get the assertion of the theorem. Consider first the case $\beta_0<n$ and $\beta_1<n$. Let $\varepsilon>0$, be such that $\beta_0+\varepsilon<n$ and $\beta_1+\varepsilon<n$. By contradiction, let there exist $x_1^\theta,\dots,x_n^\theta\in B_{A_\theta}$ such that $\|\sum_{k=1}^{n}\varepsilon_k x_k^\theta\|>B_n$ for every $\varepsilon_1, ..., \varepsilon_n = \pm 1$. As in the proof of Casini [9], for fixed $\eta > 0$ and $k = 1, \ldots, n$ we note that there exist functions $f_k \in F(\bar{A})$ such that $f_k(\theta) = \frac{x_k^\theta}{1+\eta} = x_k'$ and

$$\|f_k\| = \max_{j=0,1}(\sup_{t\in\mathbb{R}} \|f_k(j+it)\|_{A_j} \leq 1.$$

For $j = 0, 1$ and every choice of $\varepsilon_k = \pm 1$ we define

$$E^j_{\varepsilon_1,\ldots,\varepsilon_n} = \{t \in \mathbb{R} : \|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(j+it)\|_{A_j} < \frac{\beta_j + \varepsilon}{n}\}.$$

From the inequality

$$\log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k x_k' \right\|_\theta$$
$$\leq \int_{-\infty}^{\infty} \log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it) \right\|_{A_0} \mu_0(\theta,t)\,dt + \int_{-\infty}^{\infty} \log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(1+it) \right\|_{A_1} \mu_1(\theta,t)\,dt$$

where $\mu_j(\theta, t),\ j = 0, 1$ give the Poisson kernel for the strip, we obtain

$$\log \frac{\frac{B_n}{n}}{1+\eta}$$
$$< \int_{E^0_{\varepsilon_1,\ldots,\varepsilon_n}} \log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it) \right\|_{A_0} \mu_0(\theta,t)\,dt + \int_{E^{0,c}_{\varepsilon_1,\ldots,\varepsilon_n}} \log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it) \right\|_{A_0} \mu_0(\theta,t)\,dt$$
$$+ \int_{E^1_{\varepsilon_1,\ldots,\varepsilon_n}} \log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it) \right\|_{A_1} \mu_1(\theta,t)\,dt + \int_{E^{1,c}_{\varepsilon_1,\ldots,\varepsilon_n}} \log \left\| \frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it) \right\|_{A_1} \mu_1(\theta,t)\,dt.$$

Since we have $\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(j+it)\|_{A_j} \leq 1,\ j = 0, 1$ for every $t \in \mathbb{R}$, we get

$$\log \frac{\frac{B_n}{n}}{1+\eta} < (1-\theta)|E^0_{\varepsilon_1,\ldots,\varepsilon_n}| \frac{\beta_0+\varepsilon}{n} + \theta|E^1_{\varepsilon_1,\ldots,\varepsilon_n}| \frac{\beta_0+\varepsilon}{n}$$

Since η is an arbitrary positive number we have

$$\frac{B_n}{n} \leq \left(\frac{\beta_0+\varepsilon}{n}\right)^{(1-\theta)|E^0|} \left(\frac{\beta_1+\varepsilon}{n}\right)^{\theta|E^1|},$$

where $E^j = E^j_{\varepsilon_1,\ldots,\varepsilon_n}$. Replacing B_n we get

$$\left(\frac{\beta_0+\varepsilon}{n}\right)^{(1-\theta)(\frac{q}{2^n}-|E^0|)} \left(\frac{\beta_1+\varepsilon}{n}\right)^{\theta(\frac{q}{2^n}-|E^1|)} \leq 1.$$

At least one of the multipliers should be ≤ 1, let for instance this be the first one. Then since $\beta_0 + \varepsilon \leq n$ we get $|E^0| \leq \frac{q}{2^n}$. Then $|\bigcup E^0_{\varepsilon_1,\ldots,\varepsilon_n}| \leq \frac{q2^n}{2^n} = q$ (the union is taken over all permutation of signs). This means that

$$\left(\bigcup E^0_{\varepsilon_1,\ldots,\varepsilon_n}\right)^c \neq \emptyset,$$

i.e. there exist t_θ such that for every choice of signs $\varepsilon_1, \dots, \varepsilon_n$ we have

$$\|\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k f_k(it_\theta)\|_{A_0} \geq \frac{\beta_0+\varepsilon}{n}.$$

This leads us to the inequality

$$\max_{f_k}\min_{\varepsilon_k=\pm1}\|\sum_{k=1}^{n}\varepsilon_k f_k(it_\theta)\|_{A_0} \geq \beta_0+\varepsilon$$

which gives a contradiction.

If one of $\beta_0,\ \beta_1$ is equal to n, then the proof goes similarly. If for example $\beta_1 = n$ we consider only the sets $E^0_{\varepsilon_1,\dots,\varepsilon_n}$, and we use that $\log\|\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k f_k(1+it)\|_{A_j} \leq 0$ for every $t \in \mathbb{R}$.

If both $\beta_0 = n, \beta_1 = n$ the result is obvious. The proof is complete. □

Theorem 3.15. *Let $n \geq 2$, A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, $b \in \mathbb{R}\setminus\{0\}$, $0 < \theta < 1$, $1 < q < \infty$, $t = \min\{q, q'\}$, $A = A_\theta(logA)_{b,q}$ and J be the integer from the definitions of $A = A_\theta(logA)_{b,q}$.*

i) If $b < 0$, then

$$\frac{J_n(A)}{n} \leq \left\{1-\frac{1}{2^{n-1}n}\left[1-\left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^{n-1}}}\left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^{n-1}}}\max\left\{1,\left(\frac{J_n(A_1)}{J_n(A_0)}\right)^{2^{-J-n+1}}\right\}\right]\right\}^{1/t'},$$

ii)If $b > 0$, then

$$\frac{J_n(A)}{n} \leq \left\{1-\frac{1}{2^{n-1}n}\left[1-\left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^{n-1}}}\left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^{n-1}}}\max\left\{1,\left(\frac{J_n(A_0)}{J_n(A_1)}\right)^{2^{-J-n+1}}\right\}\right]\right\}^{1/t'}.$$

Proof. As a Corollary from [43, Theorem 4] we get

$$K^n_{2,2}(A) \leq 2^{\frac{1-n}{2}}\left[2^{n-1}(n-1)+c_n\right]^{1/2},$$

where $c_n = ([J^s_n(A)-n+1]_+)^2 + 2^{n-1} - 1 \leq (J_n(A)-n+1)^2 + 2^{n-1} - 1$.
Since $J_n(A) \leq n$ we have $J_n(A) - n + 1 \leq \frac{J_n(X)}{n}$ and we get

$$K^n_{2,2}(A) \leq 2^{\frac{1-n}{2}}\left[2^{n-1}n-1+\left(\frac{J_n(A)}{n}\right)^2\right]^{1/2}.$$

Since $(K^n_{2,2}(A))^2 = C^n_{NJ}(A)$, we get from [43, Theorem 4] the inequality

$$\frac{J^2_n(A)}{n} \leq C^n_{NJ}(A) \leq n - 2^{1-n} + \frac{J^2_n(A)}{2^{n-1}n^2}.$$

When $b < 0$ we prove that

$$\frac{J^2_n(A)}{n} \leq C^{(n)}_{NJ}(A) = C^{(n)}_{NJ}(\Delta_q((A_{\theta_j})_{j\in\mathbb{Z}})) \leq n^{2/t-1}\sup_{j\geq J}(C^{(n)}_{NJ}(A_{\theta_j}))^{2/t'} \leq$$

$$\leq n^{2/t-1}\left[n-2^{1-n}+\sup_{j\geq J}\frac{J_n^2(A_{\theta_j})}{2^{n-1}n^2}\right]^{2/t'}.$$

For $\theta_j=\theta+2^{-j},\ j\geq J,\ j,J\in\mathbb{N}$ such that $\theta+2^{-J}<1$ and $A_{\theta_j}=[A_0,A_1]_{\theta_j}$, using idea of the proof, similar to the idea of Theorem 3.9 we can show that

$$\sup_{j\geq J}\frac{J_n(A_{\theta_j})}{n}\leq\left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^n}}\left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^n}}\max\left\{1,\left(\frac{J_n(A_1)}{J_n(A_0)}\right)^{2^{-J-n}}\right\}.$$

The proof for the case $b>0$ is similar. □

Corollary 3.16. *Let A_0, $A_1\,\theta$, q and b be as in Theorem 3.15. If one of the spaces A_0 and A_1 is uniformly non- ℓ_n^1, then the logarithmic space $A_\theta(logA)_{b,q}$ is uniformly non- ℓ_n^1.*

Proof. A space X is uniformly non- ℓ_n^1 iff $J_n(X)<n$. So, $J_n(A_0)<n$ or $J_n(A_1)<n$. Therefore, from Theorem 3.9 we obtain $\frac{J_n(A_\theta(logA)_{b,q})}{n}<1$. Thus the space $A_\theta(logA)_{b,q}$ is uniformly non- ℓ_n^1.

Corollary 3.17. *Let A_0, $A_1\ \theta$, q and b be as in Theorem 3.15. If one of the spaces A_0 and A_1 is B- convex, then the logarithmic space $A_\theta(logA)_{b,q}$ is B-convex.*

About the classical James constant $J(X)$, using the new nice estimate $C_{NJ}(X)\leq J(X)$, [67], we get a sharper and simpler estimate of $J(A_\theta(logA)_{b,q})$.

Theorem 3.18. *Let A_0, $A_1\ \theta$, q, b, t, A and J be as in Theorem 3.15.*
i) If $b<0$, then

$$\frac{J(A_\theta(logA)_{b,q})}{2}\leq\left(\frac{J(A_0)}{2}\right)^{\frac{1-\theta}{4t'}}\left(\frac{J(A_1)}{2}\right)^{\frac{\theta}{4t'}}\max\left\{1,\left(\frac{J(A_1)}{J(A_0)}\right)^{\frac{2^{-J-2}}{t'}}\right\}$$

ii) If $b>0$, then

$$\frac{J(A_\theta(logA)_{b,q})}{2}\leq\left(\frac{J(A_0)}{2}\right)^{\frac{1-\theta}{4t'}}\left(\frac{J(A_1)}{2}\right)^{\frac{\theta}{4t'}}\max\left\{1,\left(\frac{J(A_0)}{J(A_1)}\right)^{\frac{2^{-J-2}}{t'}}\right\}.$$

3.5. **On Clarkson inequality, type and cotype of Edmunds-Triebel spaces.** Let X be a Banach space, $1<p\leq 2$ and p' the conjugate number of p, that is $\frac{1}{p}+\frac{1}{p'}=1$. X satisfies the (p,p') Clarkson inequality if

$$(\| x+y\|^{p'}+\| x-y\|^{p'})^{\frac{1}{p'}}\leq 2^{\frac{1}{p'}}(\| x\|^p+\| y\|^p)^{\frac{1}{p}}$$

for every $x,y\in X$. It is obvious that the Clarkson inequality for $p=1$ holds in every Banach space.
When we write $CI(p,p')$ holds in X, we mean that the space X satisfies the (p,p')

Clarkson inequality.

For the proof of the next proposition one can look at [31], Lemma 3.3.

Proposition 3.19. *Let X be a Banach space. If $CI(p,p')$ holds in X for some p, $1 \le p \le 2$, then $CI(r,r')$ holds in X for every r with $1 < r \le p$.*

Lemma 3.20. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of Banach spaces and $1 \le q \le \infty$. If $CI(p_n,p'_n)$ holds in X_n, for every n, where $1 \le p_n \le 2$, then $CI(p,p')$ holds in $X = [\sum_{n=1}^{\infty} \oplus X_n]_q$ for every p, $1 \le p \le \inf\{\{p_n : n \in \mathbf{N}\} \cup \{q,q'\}\}$.*

Proof. Let p, $1 \le p \le \inf\{\{p_n : n \in \mathbf{N}\} \cup \{q,q'\}\}$. Then $1 \le p \le 2$, $q \le p'$ and $CI(p,p')$ holds in X_n for every n, according to Proposition 3.19. The result is obvious for $p=1$. So we suppose that $1<p$. Let $x=(x_n), y=(y_n) \in X$. Then, by the fact that $p \le q \le p'$, Minkowski's inequality and Minkowski's reversed inequality, we have the next, with $\| x \| := \| x \|_q$:

$$\|x+y\|^{p'} + \|x-y\|^{p'} = \left(\sum_{n=1}^{\infty} \| x_n + y_n \|^q\right)^{\frac{p'}{q}} + \left(\sum_{n=1}^{\infty} \| x_n - y_n \|^q\right)^{\frac{p'}{q}}$$

$$= \left[\sum_{n=1}^{\infty} \left(\| x_n + y_n \|^{p'}\right)^{\frac{q}{p'}}\right]^{\frac{p'}{q}} + \left[\sum_{n=1}^{\infty} \left(\| x_n - y_n \|^{p'}\right)^{\frac{q}{p'}}\right]^{\frac{p'}{q}}$$

$$\le \left[\sum_{n=1}^{\infty} \left(\| x_n + y_n \|^{p'} + \| x_n - y_n \|^{p'}\right)^{\frac{q}{p'}}\right]^{\frac{p'}{q}} \le \left[\sum_{n=1}^{\infty} \left(2^{\frac{1}{p'}} (\| x_n \|^p + \| y_n \|^p)^{\frac{1}{p}}\right)^q\right]^{\frac{p'}{q}}$$

$$= \left[\sum_{n=1}^{\infty} 2^{\frac{q}{p'}} (\| x_n \|^p + \| y_n \|^p)^{\frac{q}{p}}\right]^{\frac{p'}{q}} = 2\left[\sum_{n=1}^{\infty} (\| x_n \|^p + \| y_n \|^p)^{\frac{q}{p}}\right]^{\frac{p'}{q}}$$

$$\le 2\left[\left(\sum_{n=1}^{\infty} \| x_n \|^q\right)^{\frac{p}{q}} + \left(\sum_{n=1}^{\infty} \| y_n \|^q\right)^{\frac{p}{q}}\right]^{\frac{p'}{p}} = 2(\| x_n \|^p + \| y_n \|^p)^{\frac{p'}{p}}.$$

So the proof is complete.

Lemma 3.21. *Let $0 \le \theta \le 1$ and let (A_0, A_1) be a couple of complex Banach spaces, with $A_0 \subset A_1$. If $CI(p_i,p'_i)$ holds in A_i for $i=0,1$, where $1 \le p_0, p_1 \le 2$, then $CI(p_\theta, p'_\theta)$ holds in the interpolation space $[A_0,A_1]_\theta$, where*
$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. If $p_0 = p_1 = 1$ then $p_\theta = 1$ and the result is obvious. If at least one of p_0, p_1 is bigger than 1 we consider the bounded operator

$$T : [\ell_2^{p_0}(A_0) \oplus \ell_2^{p_1}(A_1)]_1 \mapsto \left[\ell_2^{p_0'}(A_0) \oplus \ell_2^{p_1'}(A_1)\right]_\infty$$

defined by $T((x_1, x_2), (y_1, y_2)) = ((x_1 + x_2, x_1 - x_2), (y_1 + y_2, y_1 - y_2))$.
Then $\|T|_{\ell_2^{p_i}(A_i)}\| = 2^{\frac{1}{p_i'}}$, for $i = 0, 1$. Since $[\ell_2^{p_0}(A_0), \ell_2^{p_1}(A_1)]_\theta = \ell_2^{p_\theta}[A_0, A_1]_\theta$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\left[\ell_2^{p_0'}(A_0), \ell_2^{p_1'}(A_1)\right]_\theta = \ell_2^{p_\theta'}[A_0, A_1]_\theta$, where $\frac{1}{p_\theta'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}$ (see [6]), we obtain that $T\left(\ell_2^{p_\theta}[A_0, A_1]_\theta\right) \subset \ell_2^{p_\theta'}[A_0, A_1]_\theta$ and $\|T|_{\ell_2^{p_\theta}[A_0,A_1]_\theta}\| = 2^{\frac{1}{p_\theta'}}$. So $CI(p_\theta, p_\theta')$ holds in $[A_0, A_1]_\theta$ and the Lemma is proved.

Theorem 3.22. *Let A_0, A_1 be two complex Banach spaces such that $A_0 \subset A_1$. We suppose that $CI(p_i, p_i')$ holds in A_i for $i = 0, 1$, where $1 \le p_0, p_1 \le 2$. Then, whenever $0 < \theta < 1, 1 \le q \le \infty$ and $b < 0$, if $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $p = \min\{p_\theta, q, q'\}$ we have the following:*
i)If $p_0 \le p_1$ or $p_0 > p_1$ and $p < p_\theta$, then $CI(p, p')$ holds in $A_\theta(logA)_{b,q}$.
ii)If $p_0 > p_1$ and $p = p_\theta$, then for every r, $1 \le r < p$, there exists an equivalent norm in $A_\theta(logA)_{b,q}$ such that $CI(r, r')$ holds in this space.

Proof. Let $0 < \theta < 1, 1 \le q < \infty$ and $b < 0$. From Lemma 3.21 it follows that $CI(p_{\theta(j)}, p_{\theta(j)}')$ holds in $[A_0, A_1]_{\theta(j)}$, where $j \in \mathbf{N}$ is such that $\theta(j) = \theta + 2^{-j} < 1$ and $\frac{1}{p_{\theta(j)}} = \frac{1-\theta(j)}{p_0} + \frac{\theta(j)}{p_1}$.
i)If $p_0 \le p_1$, since $\theta_j \ge \theta$ we have $p_{\theta(j)} \ge p_\theta \ge p$. If $p_0 > p_1$ and $p < p_\theta$, then $p_{\theta(j)} \le p_\theta$, but since $p_{\theta(j)} \to p_\theta$ for j big enough we have $p_{\theta(j)} \ge p$.
Let $J \in \mathbf{N}$ be such that $\theta(j) < 1$ and $p_{\theta(j)} \ge p$ for every $j \ge J$. Then, according to Lemma 3.20, we obtain that $CI(p, p')$ holds in $[\sum_{j=J}^{\infty} \oplus A_{\theta(j)}]_q$, where in $A_{\theta(j)}$ we consider the norm $2^{jbq} \| \,.\, \|_{A_{\theta(j)}}$, so $CI(p, p')$ holds in $A_\theta(logA)_{b,q}$ with the norm which is defined by J.
ii)Let $1 \le r < p$ and $J \in \mathbf{N}$ be such that $\theta(j) < 1$ and $p_{\theta(j)} \ge r$ for $j \ge J$. Then the result follows by using Proposition 3.19 and Lemma 3.20 as in case (i).

The corresponding result for the case $b > 0$ reads:

Theorem 3.23. *Let $A_0, A_1, p_0, p_1\theta, q, p_\theta$ be as in Theorem 3.22. Then, for every $b > 0$, we have the following:*
i)If $p_1 \le p_0$ or $p_0 < p_1$ and $p < p_\theta$, then $CI(p, p')$ holds in $A_\theta(logA)_{b,q}$.
ii)If $p_0 < p_1$ and $p = p_\theta$, then for every $1 \le r < p$ there exists an equivalent norm in $A_\theta(logA)_{b,q}$ such that $CI(r, r')$ holds in this space.

Proof. Let $0<\theta<1, 1\leq q<\infty$ and $b>0$. From Lemma 3.21 we obtain that $CI(p_{\eta(j)}, p'_{\eta(j)})$ holds in $[A_0, A_1]_{\eta(j)}$, where $j\in \mathbf{N}$ is such that $\eta(j)=\theta-2^{-j}<0$ and $\frac{1}{p_{\eta(j)}}=\frac{1-\eta(j)}{p_0}+\frac{\eta(j)}{p_1}$.
i) Let $J\in\mathbf{N}$ be such that $\eta(j)>0$ and $p_{\eta(j)}\geq p$ for every $j\leq J$. We consider the space $A_\theta(logA)_{b,q}$ with the norm which is defined by J. Let $x, y\in A_\theta(logA)_{b,q}$ and $\epsilon>0$. Then representations $x=\sum\limits_{j=J}^{\infty}x_j$ and $y=\sum\limits_{j=J}^{\infty}y_j$ exist such that $x_j, y_j\in A_{\eta(j)}$ and

$$(1+\epsilon)\parallel x\parallel\geq\left(\sum_{j=J}^{\infty}2^{jbq}\parallel x_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{1}{q}}, (1+\epsilon)\parallel y\parallel\geq\left(\sum_{j=J}^{\infty}2^{jbq}\parallel y_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{1}{q}}.$$

Moreover

$$\parallel x+y\parallel\leq\left(\sum_{j=J}^{\infty}2^{jbq}\parallel x_j+y_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{1}{q}} \text{ and } \parallel x-y\parallel\leq\left(\sum_{j=J}^{\infty}2^{jbq}\parallel x_j-y_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{1}{q}}.$$

From Lemma 3.20 we obtain that $CI(p,p')$ holds in $\left[\sum\limits_{j=J}^{\infty}\oplus A_{\eta(j)}\right]_q$, where in $A_{\eta(j)}$ we consider the norm $2^{jbq}\parallel . \parallel_{A_{\eta(j)}}$, so we conclude that

$$\left[\left(\sum_{j=J}^{\infty}2^{jbq}\parallel x_j+y_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{p'}{q}}+\left(\sum_{j=J}^{\infty}2^{jbq}\parallel x_j-y_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{p'}{q}}\right]^{\frac{1}{p'}}$$

$$\leq 2^{\frac{1}{p'}}\left[\left(\sum_{j=J}^{\infty}2^{jbq}\parallel x_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{p}{q}}+\left(\sum_{j=J}^{\infty}2^{jbq}\parallel y_j\parallel^q_{A_{\eta(j)}}\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}.$$

So we have
$(\parallel x+y\parallel^{p'}+\parallel x-y\parallel^{p'})^{\frac{1}{p'}}\leq(1+\epsilon)2^{\frac{1}{p'}}(\parallel x\parallel^p+\parallel y\parallel^p)^{\frac{1}{p}}$ for every $\epsilon>0$.
Thus
$(\parallel x+y\parallel^{p'}+\parallel x-y\parallel^{p'})^{\frac{1}{p'}}\leq 2^{\frac{1}{p'}}(\parallel x\parallel^p+\parallel y\parallel^p)^{\frac{1}{p}}$.
ii) Let $1\leq r<p$. If $J\in\mathbf{N}$ is such that $\eta(j)>0$ and $p_{\eta(j)}\geq r$ for $j\geq J$ the result follows by using the Proposition 3.19 and Lemma 3.20 and arguing as in case (i).

For the Zygmund spaces we have the following result:

Theorem 3.24. *Let Ω be a bounded and open subset of $\mathbf{R}^n$ with Lebesgue n-measure $\mu(\Omega)<\infty, 1\leq p<2$ and $b\in\mathbf{R}$.*
i)If $b<0$ and $1\leq r<p$, then there exists an equivalent norm in $L_p(logL)_b(\Omega)$ such that this space satisfies $CI(r,r')$
ii)If $b>0$, then there exists an equivalent norm in $L_p(logL)_b(\Omega)$ such that this space satisfies $CI(p,p')$.

The proof can be found in [56], but let us note that according to standard complex interpolation we know that $[L_{p_1}, L_{p_2}] = L_p$, where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Therefore it is easy to see for instance that at least for the case $n = 1$ the statement in (i) of this theorem also follows from the preprewious theorem (ii), if $A_0 = L_2$, $A_1 = L_1$ and $\theta = \frac{2}{p+1}$.

The following information follows at once by combining the prewious theorems with the facts about relations between type, cotype and Clarkson inequalities.

Corollary 3.25. *Let $A_0, A_1, p_0, p_1, \theta, q, b, p_\theta$ and p be as in Theorem 3.22. Then we have the following:*
i)If $p_0 \leq p_1$ or $p_0 > p_1$ and $p < p_\theta$, then the spaces $A_\theta(logA)_{b,q}$ and their duals are of Rademacher type p with type constant 1 and of Rademacher cotype p' with cotype constant 1.
ii)If $p_0 > p_1$ and $p = p_\theta$, then if $1 \leq r < p$ there exists an equivalent norm in $A_\theta(logA)_{b,q}$ such that these spaces and their duals are of Rademacher type r with type constant 1 and of Rademacher cotype r' with cotype constant 1,

Corollary 3.26. *Let $A_0, A_1, p_0, p_1, \theta, q, b, p_\theta$ and p be as in Theorem 3.23. Then we have the following:*
i)If $p_1 \leq p_0$ or $p_0 < p_1$ and $p < p_\theta$, then the spaces $A_\theta(logA)_{b,q}$ and their duals are of Rademacher type p with type constant 1 and of Rademacher cotype p' with cotype constant 1.
ii)If $p_0 < p_1$ and $p = p_\theta$, then for every $1 \leq r < p$ there exists an equivalent norm in $A_\theta(logA)_{b,q}$ such that these spaces and their duals are of Rademacher type r with type constant 1 and of Rademacher cotype r' with cotype constant 1.

Corollary 3.27. *Let Ω, p and b be as in Theorem 3.24.*
i)If $b > 0$, then a renorming of the spaces $L_p(logL)_b(\Omega)$ can be made such that in this norm (in these norms) these spaces and their duals are of Rademacher type p with type constant 1 and of Rademacher cotype p' with cotype constant 1.
ii)If $b < 0$, then a renorming of the spaces $L_p(logL)_b(\Omega)$ can be made such that in this norm (in these norms) these spaces and their duals are of Rademacher type r with type constant 1 and of Rademacher cotype r' with cotype constant 1, for every $1 \leq r < p$.

A. Kaminska and B. Turett (1990) give a characterization of the Young functions ϕ so that the Orlicz space L_ϕ has type p. By using this result with ϕ equal to a Young function equivalent to $\phi(u) = u^p log^{bq}(2 + |u|)$ at infinity we obtain the same information as in the last Corollary except the precise information about

the type or cotype constants which can be obtained by making a renorming of the spaces. Moreover, this result also shows that the statements in our Corollary are in a sense the best possible. For example the following holds:
i) If $1 \leq p < 2$ and $b > 0$, then the spaces $L_p(logL)_b$ are not of type r for any $r > p$.
ii) If $1 \leq p < 2$ and $b < 0$, then the spaces $L_p(logL)_b$ are not of type p.

Another special case of the Edmunds-Triebel logarithmic spaces - the discrete spaces $l_p(logl)_b$ can be defined and similar problems can be considered. We will not give details, let us mention for instance, that if $1 < p_1 < p < p_2 < 2$ and $b > 0$, then

$$l_1 \subset l_{p_1} \subset l_p(logl)_b \subset l_p \subset l_p(logl)_{-b} \subset l_{p_2} \subset l_2.$$

We can say that the spaces $l_{p_1}, l_p(logl)_b, l_p, l_p(logl)_{-b}$ and l_{p_2} have the Rademacher type p_1, s for $\forall s < p, p, p$ and p_2, respectively.

Similar statements for the notion of cotype of the spaces $L_p(logL)_b$ and $l_p(logl)_b$ of course, can be made.

References

[1] C.Angosto, B.Cascales, Measures of weak noncompactness in Banach spaces, Topology and Applications(2008)

[2] I.Asekritova, N.Krugljak, On equivalence of K- and J-methods for $n+1-$tuples of Banach spaces, Studia Math. 122(1997), 99-116.

[3] I.Asekritova, N.Krugljak, L.Maligranda, L.Nikolova, L.-E.Persson, Lions -Peetre reiteration formulas for triples applications, Studia Math.145(3), 2OO1.

[4] A.G.Askoy and L.Maligranda, Real interpolation and measure of weak noncompactness, Math.Nachr.17591995),5-12.

[5] K.Astala, H.O.Tylli, Seminorms related to weak compactness and to Tauberian operators, Math.Proc.Cambridge Philos.Soc.107(2)(1990),36-375.

[6] J.Bergh, J.Lofstrom, Interpolation spaces.An introduction. Springer Verlag, Berlin-Heiderberg-NewJork, 1976.

[7] A. Betiuk-Pilarska, S.Phothib, S. Prus, James constant for interpolation spaces, JMAA, V.382, Is.1, Oct.2011, 127-131.

[8] F. S. Blasi, On a property of the unit shpere in a Banach space, Colloq. Math. 65 (1992), 333-343.

[9] E. Casini, Complex interpolation and l_n^1 property, Mathematica Pannonica, 3/1 (1992), 117-119.

[10] M.J.Carro, Real interpolation for families of Banach spaces, Studia Math.109(1)1994.

[11] M.J.Carro, Real interpolation for families of Banach spaces II, Collect.Math.45(1994), 53-83.

[12] M.J.Carro, Nikolova L., Interpolation of limited and weakly compact operators, Acta Appl.Math. 49:151-177, 1997.

[13] M.J.Carro, Nikolova L., Real interpolation for families of Banach spaces and compactness property, in Proceed. Of conference on function spaces in Poznan, 1998.

[14] M.J.Carro, Nikolova L., Peetre J., Persson L.E., Some real interpolation methods for families of Banach spaces - a comparision, J.Approx.Theory, 1997, 88.

[15] M.J.Carro, Nikolova L., Persson L.E., Norms of interpolation operator controlled by the Dicesar function in "Function Spaces and Applications", D.E.Edmunds et al(Eds), 2000 Narosa publ.House, New Delhi, India

[16] M.J.Carro, Peetre J., Some compactness results in real interpolation method for families of Banach spaces, J. London Math. Soc. (2) 58 (1998) 451-466.

[17] F. Cobos and A. Martinez, Remarks on interpolation properties of the measures of weak non-compactness and ideal variations, Math. Nachr. 208 (1999), 93-100.

[18] F.Cobos, Peetre J., Interpolation of compact operators: the multidimensional case, Proc.London Math.Soc. 63 (1991), 371-400.

[19] R.R. Coifman, M.Cwikel, R.Rochberg, Y.Sagher, G.Weiss, A theory of complex interpolation for families of Banach spaces, Adv.in.Math.43(1982)203-229.

[20] H. Cui, F. Wang, Gao's constants of Lorentz sequence spaces, Soochow Journal of Mathematics, 3(2007), No. 4, 707–717.

[21] M.Cwikel, S.Janson, Real and complex interpolation methods for finite and infinite families of Banach spaces, - Adv. in Math., 66, 1987, 234-290.

[22] M.Cwikel, S.Reisner, Interpolation of uniformly convex spaces, Proc. Amer. Math. Soc. 84(1982)555-559.

[23] V.I. Dmitriev and S.G. Krein, Interpolation of operators of weak type Anal. Math. vol 4 (1978), pages 83–99

[24] V.Ferenczi, A uniformly convex hereditarily indecommpossable banach spaces, Isr.J.of Math. Vol.102, No1(1997) 199-225.

[25] D.Edmunds and H.Triebel, Logarithmic spaces and related trace problems, Funct.Approximatio, XXVI(1998), 189-204.

[26] D.L.Fernandez, Interpolation of 2^n Banach spaces, Studia Math.65(1979), 175-201.

[27] T.Figiel, On the moduli of convexity and smoothness, Studia Math.LVI(1976), 121-155.

[28] J. Gao, A Pythagorean approach in Banach spaces, J. Inequal. Appl. 2006, Article ID 94982 (2006), 1–11.

[29] J.E.Gilbert, Interpolation between weighted L^p spaces, Ark.Mat.10(1972), 235-249.

[30] M. Kato, L. Maligranda, Y. Takahashi, On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces, Studia Math. 144 (2001), 275–295.

[31] Kato M., Persson L.E. and TakahashiY.: Clarkson type inequalities and their relations to the concepts of type and cotype, Collect. Math. 51 (2000), 327-346.

[32] Kato M. and Takahashi Y.: Type, cotype constants and Clarksons inequalities for Banach spaces, Math. Nachr. 186(1997), 187-348.

[33] M. Kato, Y. Takahashi and K. Hashimoto, On n-th von Neumann-Jordan constants for Banach spaces, Bull. Kyushu Inst. Tech. Pure Appl. Math. 45 (1998), 25-33.

[34] S.G.Krein, L.I.Nikolova, Holomorphic functions in a family of Banach spaces, Interpolation, Dokl.Acad Nauk SSSR(250)1980, 547-550. Engl.tr. Soviet Math.Dokl.21, 1980, 131-134.

[35] S.G.Krein, L.I.Nikolova, Complex method of interpolation for family of Banach spaces, Ukrain Mat.Z.1982, 32942. Engl.tr. Ukrain.Mat.J.34, 1982, 26-36

[36] S.G.Krein, Yu.I.Petunin, E.M.Semenov, Interpolation of linear operators, (Russian), M.Nauka, 1978, 400pg.

[37] A. Kryczka, Weak noncompactness in Banach sequence spaces and its extrapolation properties, Math. Inequal. and Appl. 11 (2008), 297-306.

[38] A. Kryczka and S. Prus, Measure of weak noncompactness under complex interpolation, Studia Math. 147 (2001), 89-102.

[39] A. Kryczka, S. Prus and M. Szczepanik, Measure of weak noncompactness and real interpolation of operators, Bull. Austral. Math. Soc. 62 (2000), 389-401.

[40] D.Kutzarova, Nikolova L., Prus S., Infinite dimensional geometric properties of interpolation spaces, Math. Nachr.191, 1998, 215-228.

[41] D.Kutzarova , L.Nikolova, Zachariades T., Real interpolation for families of Banach spaces and convexity, Math. Nachr.171, 1995, 259-268.

[42] J.Lindenstrauss, L.Tzafriri, Classical Banach Spaces II, Springer-Verlag, 1979.

[43] L.Maligranda, Nikolova L., Persson L.-E., Zachariades T., On n-th James and Khintcin constants of Banach spaces, Math.Inequal. Appl. V.11, No1, 2008.

[44] K. I. Mitani, K. Saito, The James constant of absolute norms on R^2, Journal of Nonlinear and Convex Analysis, 4(2003), No 3, 399–410.

[45] H.Mizuguchi, K-S.Saito, Some geometric constants of absolute normalized norms on R^2, Ann. Funct. Anal.2(2011), No2, 22-33.

[46] L.I.Nikolova, Complex method of interpolation for several spaces, C.R.Bulg.Sci, 31, 1978, 1523-1526.(Russian)

[47] L.Nikolova, On interpolation of compactness property in families of Banach spaces, Research.rep.1990-14, ISSN-1101-1327, lulea, Sweden.

[48] L.I.Nikolova, Some estimates of measure of noncompactness for operators- the multidimensional case, C.R.Acad.Bulg.Sc.12, 44, 1991.

[49] L.I.Nikolova, On the classes K_θ and J_θ in the case of interpolation in family of Banach spaces, C.R.Acad.Bulgar.Sci.41(1988), 9-12, (Russian).

[50] L.I.Nikolova, Interpolation of some properties of operators acting in families of Banach spaces. Ann.Sofia Univ.84, 1990.

[51] L.I.Nikolova, L.E.Persson, Interpolation of nonlinear operators between families of Banach spaces, Math.Scand.72, 1993, 47-60.

[52] L. Nikolova, Persson L.E., Real interpolation methods for families of Banach spaces, Res.rep1996-23, ISSN, 1400-4003.

[53] L.Nikolova, L.E.Persson, S.Varoshanec, The Beckenbach-Dresher inequality in the ψ-direct sums of spaces and related results, J.Ineq.Appl.2012, 2012:7

[54] L. Nikolova, L.E Persson, T. Zachariades , Estimates of some constants equipped with Banach spaces, Research report, Lulea University of Technology, 12 (2003), 1-13.

[55] L. Nikolova , Persson L.-E., Zachariades T., A Study of some constants for Banach spaces, C.rend.Acad.Bul.Sci, tome 57, No2, 2004

[56] L. Nikolova , Persson L.E.Zachariades T., On Clarkson's inequality, type and cotype for the Edmunds-Triebel logarithmic spaces, Arch.Math.80(2003)165-176.

[57] L. Nikolova, Zachariades T., The uniform convexiti of the Edmunds-Triebel logarithmic spaces, J.Math.Anal.Appl 283(2003) 549-556.

[58] L. Nikolova , Zachariades T., On Edmunds-Triebel spaces, Banach J.Math.Anal. Volume 4, Number 1 (2010), 146-158.

[59] L. Nikolova, S.Varoshanec, Refinement of Holder 's inequality derived from functions $\psi_{p,q,\lambda}$. Ann.Funct.Anal., Vol.2, No1, 2011, 72-83.

[60] L. Nikolova , S.Varoshanec, Constants for Banach spaces with absolute normalized norms which corresponding functions are comparable, to appear in J.Non-conv.Anal.

[61] J.Peetre, Recent progress in real interpolation spaces, in Meth.of funct, anal.and theory of ellipt.eq., Proceedings in mem.C.Miranda, Naples, Sept, 1982.

[62] A.Pietsch, Operators Ideals, North-Holland, Amsterdam 1980.

[63] G.Pisier, Complex interpolation between Hilbert, Banach and operator spaces, Memories of AMS 208(2010), 978.

[64] K. Saito, M. Kato, Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on C^2, J. Math. Anal. Appl. 244(2000), 515–532.

[65] S. Saejung, Another look at Cesaro sequence spaces, J. Math. Anal. Appl. 366(2010) 530 537.

[66] G.Sparr, Interpolation of several Banach spaces, Ann. Mat. Pura Appl. 99(1974), 247-376.

[67] Y. Takahashi and M. Kato, A simple inequality for the von Neumann-Jordan and James constants of a Banach space, J. Math. Anal. Appl. 359 (2009), 602-609.

[68] M.F.Teixeira, D.E.Edmunds, Interpolation theory and measure of noncompactness, Math.Nachr., 104, 1981, 129-135.

[69] H.Triebel, Interpolation Theory, Function Spaces, Differential operators, North-Holland, Amsterdam 1978.

[70] A. Yoshikawa, Sur la teorie d'espaces d'interpolation- les espaces de moyenne de plusieurs espaces de Banach, J. Fac. Sci. Univ. Tokyo Sect. I 16 (1970), 407- 468.

[71] Z. Zuo, A geometrical constant and normal structure in Banach Spaces, Journal of Inequalities and Applications 2011, 2011:16, doi:10.1186/1029-242X-2011-16.