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Subdifferential analysis of convex-like functions

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Dedication

I dedicate this work to the memory of my mother, Ivanka Konstantinova who gave me a deep appreciation for mathematics and physics in my younger years. She always had high hopes for me but sadly couldn't live long enough to see them unfold.

Contents

Preface

The need for studying non-smooth functions, non-smooth optimization problems and sets with a non-smooth boundary has emerge quite natural with the development of modern mathematics. *Operations Research* and *Variational Analysis* are good examples of fields of research that study these objects. Concepts like distance functions, solution-sets, projection-sets, indicator functions, normal cones, tangent cones and subdifferentials are in the heart of the development of this fields, but all of them generally are unavoidably non-smooth. As an example consider the real valued function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := |x| = \max\{x, -x\}.$$

This function f is the maximum of two differentiable functions but it is non-differentiable at $x = 0$. Nonetheless it's graph does not have a unique tangent line at $(0, 0)$, but a whole family of tangent lines. The concept of subdifferentials comes quite in handy to characterize this family of tangent lines. In the sense of *Convex Analysis* the *convex subdifferential* $\partial f(x)$ for this function f at x is the set

$$\partial f(x) := \{p \in \mathbb{R} : f(x) + p(y - x) \leq f(y),\ \forall y \in \mathbb{R}\}.$$

One can interpret the latter as the set of all of the gradients of liner functions which are tangent to the point $(x, f(x))$ of the graph of the function f and are always under it. It can be shown that

$$\partial f(x) = \begin{cases} \{-1\}, & x < 0, \\ [-1, 1], & x = 0, \\ \{1\}, & x > 0. \end{cases}$$

and that if $p_0 \in \partial f(0)$, then the vector $(p_0, -1)$ is normal to one of the tangent lines to the graph of the function f at $(0, 0)$.

This example is not an artificial one. The maximum of two linear functions appears in some types of *Cutting Stock Problems*, which are widely used in optimization problems coming from the manufacturing industry.

For a Banach space X the convex subdifferential of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$ is the set

$$\partial f(x) := \{p \in X^* : \langle p, y - x \rangle \leq f(y) - f(x),\ \forall y \in X\}.$$

Convex subdifferentials have many properties which resembles classical properties of derivatives. One can say that they are a continuation and generalization of derivatives. For example, one has that if $0 \in \partial f(x)$ then the point x is a minimzer of the function f, the set $\partial f(x)$ is reduced to the singleton $\{f'(x)\}$ when f is differentiable at the point x, sum rules of the forme

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$$

for appropriately chosen function f and g and many more.

Of course there are other subdifferentials except the convex subdifferential. For example the *Dini* subdifferential, the *Clarke* subdifferential and the *Michel-Penot* subdifferential which definitions relay respectively on the corresponding generalized derivatives of *Dini*, *Clarke* and *Michel-Penot*, see [16, Chapter 6]. One can also define a subdifferential axiomatically as an abstract subdifferential, see [50].

Let C be a convex subset of a real valued vector space. The function $f : C \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if the following inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

holds for all $x, y \in C$ and $\lambda \in [0, 1]$.

The function f is said to be strictly convex if the latter inequality is strict for all $x, y \in C$ and $\lambda \in (0, 1)$.

The geometric interpretation of this definition is that for fixed x and y in C the graph of the function f on the segment $[x, y]$ lies below the segment between the point on the graph $(x, f(x))$ and $(y, f(y))$.

Notice that this definition does not rely on derivatives, although there are classical result of necessary and sufficient conditions for a differentiable function to be convex which include derivatives. One example is that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if its derivative f' is monotone increasing.

Another way to characterise a convex function $f : C \to \mathbb{R}$, where the set C is a convex subset of some real valued vector space, is by its epigraph which is the set

$$\operatorname{epi} f := \{(x, r) \in C \times \mathbb{R} : f(x) \leq r\}.$$

One has that f is convex if and only if it's epigraph $\operatorname{epi} f$ is a convex set in $C \times \mathbb{R}$, i.e.

$$\lambda(x_1, r_1) + (1 - \lambda)(x_2, r_2) \in \operatorname{epi} f$$

for all $(x_1, r_1), (x_2, r_2) \in \operatorname{epi} f$ and $\lambda \in [0, 1]$.

This necessary and sufficient conditions gives us an important connection between convex functions and their epigraphs and gives us yet another good geometric intuition.

There is no debate that convex functions play a key role in many optimization problems due to their wide range of convenient properties. For example if a strictly convex functions has a minimizer it is unique. However, finding new results for *convex-like* function is essential. By *convex-like* we mean functions which have properties similar to those of convex functions but are not necessarily convex. One such class of functions is the class of *primal lower-nice functions* which was introduced by Poliquin in 1991, see [41] and has been extensively studied ever since, see [42, 43, 27, 9, 52, 36]. In his work [41] Poliquin shows that *primal lower-nice functions* considered on a finite dimensional space can be fully characterized by their *Clarke* subdifferential. But there is not only one way to define *primal lower-nice functions*. For example Ivanov and Zlateva in [27] show that the following two definitions are equivalent:

Definition 1 Let $f : X \in \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Function f is said to be primal lower nice at $\bar{x} \in \operatorname{dom} f$ if there exist $\lambda > 0$, $c > 0$ and $T > 0$ such that

$$f(y) \geq f(x) + \langle p, y - x \rangle - \frac{t}{2} \|y - x\|^2,$$

where $t \geq T$, $x \in \bar{x} + \lambda\mathbb{B}, y \in x + \lambda\mathbb{B}, p \in \partial f(x)$ and $\|p\| \leq ct$.

Definition 2 Let $f : X \in \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Function f is said to be primal lower nice at $\bar{x} \in \operatorname{dom} f$ if there exist $\lambda > 0$, $c > 0$ and $T > 0$ such that

$$\langle p - q, x - y \rangle \geq -t\|x - y\|^2,$$

where $t \geq T$, $x, y \in \bar{x} + \lambda\mathbb{B}, p \in \partial f(x), q \in \partial f(y)$ and $\max\{\|p\|, \|q\|\} \leq ct$.

Afterwards Ivanov and Zlateva showed in [28] that the proximal subdifferential and the Clarke subdifferential of a *primal lower nice function* defined on a β smooth Banach space coincide. This result suggests that the class of *primal lower nice functions* does not depend on the subdifferential involved in their definition. After these results a very interesting question arises: Is it possible to characterize *primal lower nice functions* without using subdifferentials?

In Chapter 2 we show that *primal lower nice functions* defined on a Hilbert space satisfy the following property: For any $a, b \in \operatorname{dom} f$ such that

$$\sqrt{\|a - b\|^2 + (f(a) - f(b))^2} < 2r$$

and any $\lambda \in [0, 1]$ there is $u \in \operatorname{dom} f \cap B[\lambda a + (1 - \lambda)b, \varphi(\lambda)]$ such that either

$$f(u) \leq \lambda f(a) + (1 - \lambda) f(b),$$

or

$$\lambda f(a) + (1-\lambda)f(b) < f(u) \leq \lambda f(a) + (1-\lambda)f(b) + \varphi(\lambda),$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

In particular,

$$\inf_{B[\lambda a+(1-\lambda)b), \varphi(\lambda)]} f \leq \lambda f(a) + (1-\lambda)f(b) + \varphi(\lambda).$$

Note that this property does not use subdifferential in any way. To achieve it we introduce and study a property we call *epi prox-regularity* of an epigraph set which slightly differs from the well-known prox-regularity property of a set. We take this approach because *primal lower nice* functions are strongly related to *prox-regular* sets. Indeed in a Hilbert space a set is prox-regular if and only if it's indicator function is primal lower nice, see [44, Proposition 2.1].

The study of prox-regular sets takes root in the pioneering work of Federer who gave an extension of the Steiner polynomial formula for convex set by introducing *positively reached sets*, see [23]. During the years different authors have given different names for such sets. For example weakly convex [53] and proximally smooth sets [19]. Other names can be found in the survey [20].

It is key to point out that prox-regular sets and convex sets have some common properties. For example the distance function is differentiable and Lipschitz continuity on some appropriately chosen tube both for prox-regular and convex sets. But it is not true that the intersection of two prox-regular sets is prox-regular.

In Chapter 1 we provide a new proof of an intrinsic property of prox-regular sets in Hilbert spaces. The term prox-regularity was given by Poliquin and Rockafellar in [43]. They defined the term for sets and functions and unfold their properties first in $\mathbb{R}^n$. In [44, Theorem 4.1] Poliquin, Rockafellar and Thibault achieved various characterizations of a prox-regular set C defined on a Hilbert space. Note that all of this characterizations use the distance function, the projection set or the proximal normal cone in some way. The characterization which we prove does not use them and is the following:

For any $a, b \in C$ such that $\|a-b\| < 2r$ and any $\lambda \in (0,1)$ for

$$x_\lambda := \lambda a + (1-\lambda)b$$

there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}. \tag{1}$$

This characterization is well know, but the methods used in our proof help us to achieve the results in Chapter 2. This is because of the relationship between a prox-regular function and its epigraphs.

In Chapter 3 we give new proof of the Moreau-Rockafellar theorem which states that a proper, lower semicontinuous and convex function on a Banach space is determined up to a constant by its subdifferential. To this end we develop a novel Epsilon Subdifferential Method (ESM) which is similar to the classical Epsilon Subdifferential Method, see e.g. [12, 13]. For convenience of the reader we layout the novel ESM here:

It applies to a given proper, convex and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined on a Banach space X, such that

$$0 = f(0) = \min_{x \in X} f(x)$$

with fixed in advance parameters $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$.

Starting at arbitrary $x_0 \in \operatorname{dom} f$, for $i = 0, 1, \ldots$

- if $0 \in \partial_\varepsilon f(x_i)$, then STOP;
- if $0 \notin \partial_\varepsilon f(x_i)$, for

  $$\varphi_{x_i}(K) := \inf_{x \in X} F_{x_i}(K, x),$$

  where

  $$F_{x_i}(K, x) := f(x) - f(x_i) + \varepsilon + K\|x - x_i\|,$$

  find $K_i > 0$ such that $\varphi_i(K_i) = 0$.

  Take any x_{i+1} satisfying

  $$0 \le f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \le \delta.$$

Although in the classical Epsilon Subdifferential Method the function under consideration is defined on $\mathbb{R}^n$ there is no issue to continue the method to the infinite dimensional case. A key difference between the novel and the classical ESM is that the classical approximates a minimum of the function under consideration by making the epsilon in the method smaller and smaller every time when $0 \in \partial_\varepsilon f(x_i)$, where x_i has been generated in the previous iteration of the method, while our novel ESM finds an ε-minimum for a fixed in advance positive ε, i.e. it stops when $0 \in \partial_\varepsilon f(x_i)$ and x_i is the last point found by it. The latter is not a disadvantage of the novel EMS, because finding a ε minimum is more than enough to prove in a new way the famous *Rockafellar-formula*, (see [45],[46] and [29, Theorem 1.2]):

Rockafellar-formula Let

$$g : X \to \mathbb{R} \cup \{+\infty\}$$

be a proper, lower semicontinuous and convex function defined on a Banach space X. Let $\bar{x} \in \operatorname{dom} \partial g$ and $\bar{p} \in \partial g(\bar{x})$. Then for all $x \in X$

$$g(x) = g(\bar{x}) + R_{\partial g,(\bar{x},\bar{p})}(x),$$

where

$$R_{\partial g,(\bar{x},\bar{p})}(x) := \sup \Big\{ \sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle : \\ x_0 = x,\ x_n = \bar{x},\ q_n = \bar{p},\ q_i \in \partial g(x_i), n \in \mathbb{N} \Big\}.$$

By proving in a new way the famous *Rockafellar-formula* we give a new prove of the *Moreau-Rockafellar Theorem*, (see [45, 46]):

Moreau-Rockafellar Theorem Let X be a Banach space. Let g and h be proper, lower semicontinuous and convex functions from X to $\mathbb{R} \cup \{+\infty\}$. If $\partial g \subset \partial h$, then $h = g + \text{const}$.

Historically the first complete proof of the famous Moreau-Rockafellar theorem in a Banach space is due to Rockafellar, see [46]. However this proof depends on duality arguments. A simpler proof which does not use any duality was done by Ivanov and Zlateva in [29] which resembles the proof that a monotone function is Riemann integrable (a classical result in Calculus). To this end they prove the *Rockafellar-formula* by using in [29] the following **Lemma 3.3** proved by Ekeland variational principle.

Lemma 3.3 Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Let $(y_i)_{i=1}^k \subset \operatorname{dom} f$. The for each $\varepsilon > 0$ there are $(x_i, p_i) \in \operatorname{gph} \partial f$ such that

$$\|x_i - y_i\| \le \varepsilon \text{ and } \|p_j\| \, \|x_i - y_i\| \le \varepsilon,\ \forall i, j = 1, \dots, k.$$

Since the proof of *Lemma 3.3* relays on Ekeland variational principle the relationship between x_i and p_i is not clear. One of the main features and merits of the novel ESM is to partially clarify and reveal the relationship between them. This is done without using a Variational principle.

With a pinch of optimism we expect our novel ESM to be useful for proving the integrability for the class of uniformly lower regular functions defined on a Hilbert space considered in Chapter 2.

For the convenience of the reader proofs of some of the routine results and well known facts are given in the Appendix.

Chapter 1

An intrinsic property of prox-regular sets in Hilbert space

The study of prox-regular sets, a term due to Poliquin, Rockafellar and Thibault [44], can be traced back to the pioneering work [23] of Federer who introduced them as positively reached sets in $\mathbb{R}^n$. During the years, various names of such sets have been introduced: weakly convex [53] or proximally smooth sets [19] are commonly used in Hilbert spaces; for other names see the survey [20]. Prox-regular sets in Banach spaces are studied in [10, 11, 6, 8].

Along with the study of prox-regular sets from a theoretical point of view, they are intensively studied and involved in the famous Moreau's sweeping processes, see the survey [39] and the references therein. Various stability and separation properties of prox-regular sets are established in [1, 3, 4]. More details one can find in the paper [44], the survey [20], the forthcoming book [51] and their bibliography.

Prox-regularity has been introduced as an important new regularity property in Variational Analysis by Poliquin and Rockafellar in [43]. They defined the concept for functions and sets and developed the subject in $\mathbb{R}^n$. Numerous significant characterizations of prox-regularity of a closed set C in Hilbert space at point $\overline{x} \in C$ were obtained by Poliquin, Rockafellar and Thibault in [44] in terms of the distance function d_C and metric projection mapping P_C, e.g. d_C being continuously differentiable outside of C on a neighbourhood of $\overline{x}$, or P_C being single-valued and norm-to-weak continuous on this same neighbourhood. On global level, in [44] the authors showed that uniformly prox-regular sets are proximally smooth sets providing new insights on them.

In this chapter we will prove the following intrinsic characteristic properties of a r-prox-regular set.

Theorem 1.1.1. *Given a real $r > 0$, a non-empty closed set C in a Hilbert space H. The following are equivalent:*

(a) C is r-prox-regular.

(b) For any $a, b \in C$ such that $\|a - b\| < 2r$ and any $\lambda \in (0,1)$ for

$$x_\lambda := \lambda a + (1-\lambda)b$$

there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \le r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}. \tag{1.1}$$

(c) For any $a, b \in C$ with $\|a - b\| < 2r$ there is some $z \in C$ such that

$$\left\|\frac{a+b}{2} - z\right\| \le r - \sqrt{r^2 - \frac{\|a-b\|^2}{4}}. \tag{1.2}$$

The equivalence (a) $\Leftrightarrow$ (c) is established by G. E. Ivanov, see [26, Lemma 4.2] by using the properties of the sets $\Delta_r(a,b) := \bigcap_{d:\{a,b\}\in B[d,r]} B[d,r]$, first considered by J.-P. Vial, see [53]. In our proof we use a different approach which does not rely on these sets.

In finite dimensional settings, J.-P. Vial, see [53, Proposition 3.4], proved the implication (a) $\Rightarrow$ (b) with right hand side of (1.1) equal to $\theta_\lambda := \dfrac{\lambda(1-\lambda)}{r}\|a-b\|^2$, and the implication (b) $\Rightarrow$ (a) with right hand side of (1.1) equal to $\delta_\lambda := \dfrac{\lambda(1-\lambda)}{2r}\|a-b\|^2$. As $\delta_\lambda < r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} < \theta_\lambda$, the condition (1.1) is slightly weaker than both conditions of Vial. The equivalence (a) $\Leftrightarrow$ (b) is proved in Hilbert settings in [51, Proposition 15.41], by using different arguments.

The results in this chapter are published in [33].

1.1 Notations

Throughout this chapter H stands for a real Hilbert space endowed with the inner product

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R},$$

and with the associated with it norm

$$\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}.$$

The open (resp. closed) ball of H centered at $x \in H$ with radius $t > 0$ is denoted by

$$B(x,t) := \{y \in H : \|y - x\| < t\} \text{ (resp. } B[x,t] := \{y \in H : \|y - x\| \le t\}).$$

In the particular case of the closed unit ball we use the notation

$$\mathbb{B} := B[0;1].$$

In the following notations we consider C to be a nonempty subset of H.

The distance function

$$d_C : H \to \mathbb{R}_+,$$

which measures the distance of a point $x \in H$ to the set C is defined as

$$d_C(x) := \inf_{y \in C} \|x - y\|, \quad \text{for all } x \in H.$$

For $\varepsilon \geq 0$ the $\varepsilon-$argmin set of the distance function is defined as

$$\varepsilon-\operatorname{argmin} d_C(x) := \{y \in C : \|x - y\| \leq d_C(x) + \varepsilon\}.$$

For an extended real $r \in (0, +\infty]$ through the distance function, one defines the (open) r-tube of C as the set

$$T_C(r) := U_C(r) \setminus C,$$

where $U_C(r)$ is the (open) r-enlargement of C

$$U_C(r) := \{x \in H : d_C(x) < r\}.$$

In Figure 1.1 are shown examples of r-tubes of sets in $\mathbb{R}^2$.

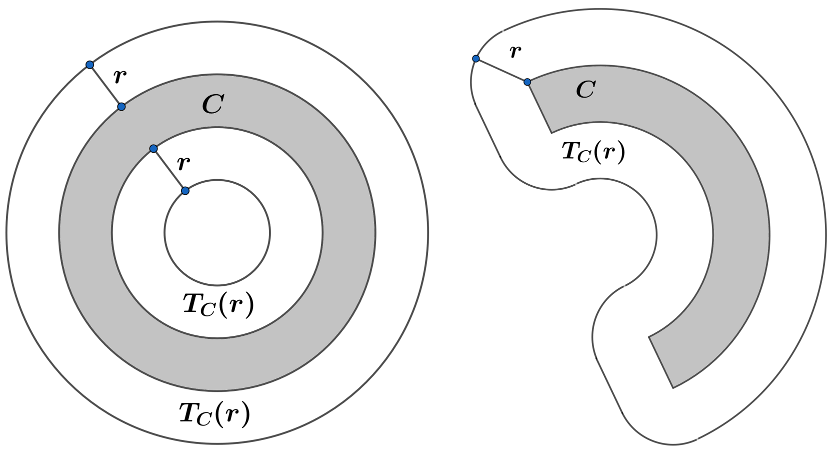

Figure 1.1: Examples of two (open) r-tubes in $\mathbb{R}^2$.

The multi-valued mapping $P_C : H \rightrightarrows H$ which gives the set of all nearest points in C to a point $x \in H$ is defined by

$$P_C(x) := \{y \in C : d_C(x) = \|x - y\|\}, \quad \text{for all } x \in H.$$

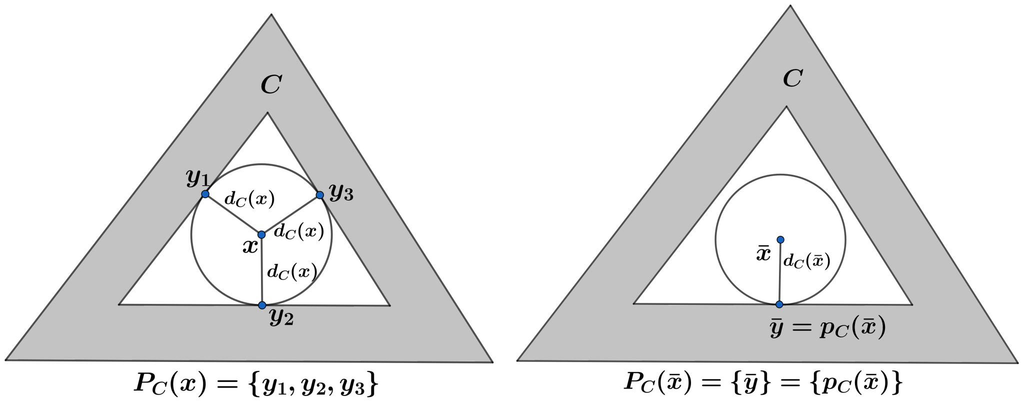

Figure 1.2: Examples of projection sets in $\mathbb{R}^2$.

The set $P_C(x)$ is called the projection set of the point x to the set C.
Whenever for some $\overline{x} \in H$ the latter set is reduced to a singleton, i.e.

$$P_C(\overline{x}) = \{\overline{y}\},$$

the vector $\overline{y} \in H$ is denoted by $p_C(\overline{x})$.

In Figure 1.2 are shown examples of multi-valued and single valued projection sets in $\mathbb{R}^2$.

The proximal normal cone of C at $x \in H$, denoted by $N_C(x)$, is defined as,(see [47])

$$N_C(x) := \{p \in H : \text{ there exist } r > 0 \text{ such that } x \in P_C(x + rp)\}.$$

By convention, $N_C(x') = \varnothing$ for all $x' \notin C$. The elements of the proximal normal cone $N_C(x)$ are called *proximal normals* to the set C at x.

It is easy to see that $p \in N_C(x)$ if and only if there is a real $\sigma > 0$ such that

$$\langle p, x' - x\rangle \leq \sigma \|x' - x\|^2, \quad \text{for all } x' \in C. \tag{1.3}$$

The proof of the latter is given in Proposition A.1.1 in the Appendix.

It is key to point out that the $\sigma > 0$ in (1.3) depends on x as well as on p.

The following definition considers such nonempty closed subset of H for which the $\sigma > 0$ in (1.3) stays the same for all proximal normals taken at $x \in C$.

Definition 1.1.2. *Let C be a nonempty closed subset of H and $r \in (0, +\infty]$. One says that C is r-prox-regular (or uniformly prox-regular with constant r) whenever, for every $x \in C$ and $p \in N_C(x) \cap \mathbb{B}$ one has that $x = p_C(x + rp)$, i.e. x is the unique nearest point from $x + rp$ to C.*

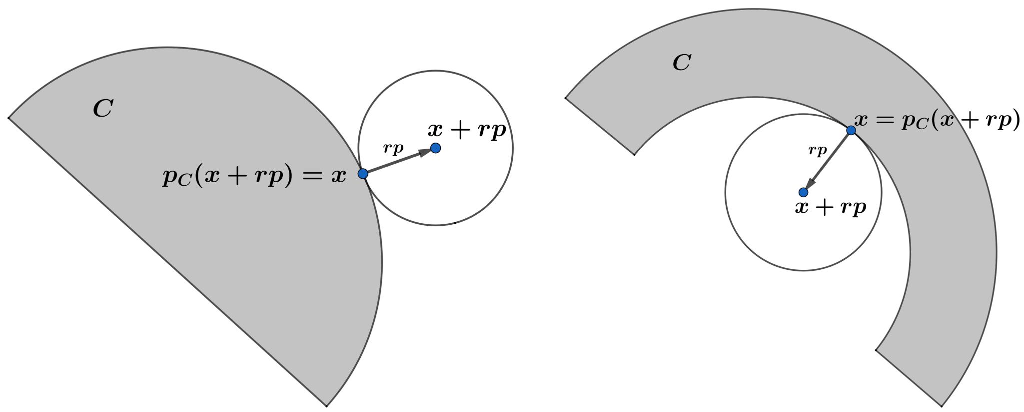

Figure 1.3: Example of two r-prox-regular sets in $\mathbb{R}^2$.

The geometric interpretation of Definition 1.1.2 is that on such a set we can roll a ball with a fixed radius $r > 0$ over its entire boundary (see Figure 1.3).

In the following theorem are collected some of the characterizations of uniformly prox-regular sets for which we refer to [44, Theorem 4.1].

Theorem 1.1.3. *Let C be a nonempty closed subset of H and let $r > 0$. The following assertions are equivalent:*
(a) The set C is r-prox-regular.
(b) For all $x, x' \in C$, for all $p \in N_C(x)$, one has

$$\langle p, x' - x\rangle \leq \frac{1}{2r}\|p\|\|x' - x\|^2.$$

(c) P_C is single-valued and norm-to-weak continuous on $T_C(r)$.

1.2 Proof of the intrinsic property

We will prove Theorem 1.1.1 by establishing the relations

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a).$$

$(a) \Rightarrow (b)$ Let C be r-prox-regular and $a, b \in C$ be such that $\|a - b\| < 2r$. Fix $\lambda \in (0, 1)$ and denote

$$x_\lambda := \lambda a + (1 - \lambda)b.$$

First we will show that $x_\lambda \in U_C(r)$. Since $a, b \in C$ by the definition of the distance function we have that

$$d_C(x_\lambda) = \inf_{y \in C} \|x_\lambda - y\| \leq \|x_\lambda - a\| = (1 - \lambda)\|a - b\|$$

and

$$d_C(x_\lambda) = \inf_{y\in C} \|x_\lambda - y\| \leq \|x_\lambda - b\| = \lambda\|a-b\|.$$

Hence we get that

$$d_C(x_\lambda) \leq \min\,(1-\lambda, \lambda)\,\|a-b\| \leq \frac{1}{2}\|a-b\| < r,$$

which shows that $x_\lambda \in U_C(r)$.If $x_\lambda \in C$ we can just take $u_\lambda = x_\lambda$ and the proof will be complete. So we will consider only the case $x_\lambda \in T_C(r)$. From Theorem 1.1.3 (c) there exists a unique $u_\lambda \in C$ such that

$$u_\lambda := p_C(x_\lambda).$$

In Figure 1.4 is given some sketch in $\mathbb{R}^2$.

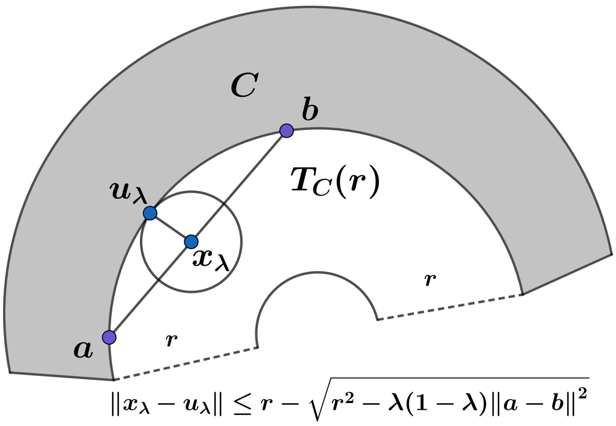

Figure 1.4: Geometric idea for the prove of $(a) \Rightarrow (b)$

Since λ is fixed, further we will omit it from the index and will work with $x := x_\lambda$, and $u := u_\lambda$ instead. Set $p := x - u$ and observe that $p \neq 0$ and that $p \in N_C(u)$.From Theorem 1.1.3 (b) it holds that

$$\langle p, x' - u\rangle \leq \frac{1}{2r}\|p\|\|x'-u\|^2, \quad \forall x' \in C. \tag{1.4}$$

It is clear that

$$u = x - p = \lambda a + (1-\lambda)b - p. \tag{1.5}$$

Substituting $x' = a$ in (1.4) and using the expression (1.5) for u, we get

$$\begin{aligned}\langle p, (1-\lambda)(a-b) + p\rangle &\leq \frac{1}{2r}\|p\|\|(1-\lambda)(a-b)+p\|^2 = \\ &= \frac{1}{2r}\|p\|\Big((1-\lambda)^2\|a-b\|^2 + 2(1-\lambda)\langle a-b, p\rangle + \|p\|^2\Big)\end{aligned} \tag{1.6}$$

Analogously, substituting $x' = b$ in (1.4) we have

$$\langle p, \lambda(b-a)+p\rangle \le \frac{1}{2r}\|p\|\Big(\lambda^2\|b-a\|^2 + 2\lambda\langle b-a, p\rangle + \|p\|^2\Big). \tag{1.7}$$

Multiplying inequality (1.6) by λ, inequality (1.7) by $(1-\lambda)$ and adding them, we obtain

$$\langle p, p\rangle \le \frac{1}{2r}\|p\|\left(\lambda(1-\lambda)\|a-b\|^2 + \|p\|^2\right).$$

Rearranging the latter, we have that $\|p\|$ satisfies the following quadratic inequality

$$t^2 - 2rt + \lambda(1-\lambda)\|a-b\|^2 \ge 0. \tag{1.8}$$

Since $\|a-b\| < 2r$, and $\lambda \in (0,1)$ for the discriminant of the left-hand side of this quadratic inequality we have

$$\begin{aligned} D &:= 4r^2 - 4\lambda(1-\lambda)\|a-b\|^2 > 4r^2 - 4\lambda(1-\lambda)4r^2 \\ &= 4r^2\big(1-4\lambda(1-\lambda)\big) = 4r^2(1-4\lambda+4\lambda^2) \\ &= 4r^2(1-2\lambda)^2 \ge 0. \end{aligned} \tag{1.9}$$

Hence any t satisfying (1.8) is such that $t \le t_1$ or $t \ge t_2$, where

$$t_1 := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2},$$

and

$$t_2 := r + \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Having in mind that $u = p_C(x)$, we have

$$\|p\| = \|x-u\| \le \|x-a\| = \|\lambda a + (1-\lambda)b - a\| = (1-\lambda)\|b-a\|,$$

and

$$\|p\| = \|x-u\| \le \|x-b\| = \|\lambda a + (1-\lambda)b - b\| = \lambda\|b-a\|.$$

Hence

$$\|p\| \le \min(1-\lambda, \lambda)\|b-a\| \le \frac{\|b-a\|}{2} < \frac{2r}{2} = r.$$

As $t_2 \ge r$, we obviously get that $\|p\| \le t_1$, which reads

$$\|p\| \le r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2},$$

and the proof of (a) $\Rightarrow$ (b) is completed.

Remark 1.2.1. *For any $\lambda \in (0,1)$ we have that $u_\lambda \notin \{a, b\}$.*

The proof is given in Proposition A.1.3 in the Appendix.

$(b) \Rightarrow (c)$ The proof is obvious, just take $\lambda = \frac{1}{2}$ in (1.1).

$(c) \Rightarrow (a)$ Let x_0 be any point in $T_C(r)$, i.e. $0 < d_C(x_0) < r$ and denote

$$\Delta := \frac{1}{2} \min \{d_C(x_0), r - d_C(x_0)\}.$$

Take an arbitrary $x \in B(x_0, \Delta)$. Note that by the choice of Δ we have that

$$d_C(x) = \inf_{y \in C} \|x - y\| \leq \inf_{y \in C} (\|x - x_0\| + \|x_0 - y\|) =$$

$$= \|x - x_0\| + \inf_{y \in C} \|x_0 - y\| = \|x - x_0\| + d_C(x_0) <$$

$$< \Delta + r - 2\Delta = r - \Delta.$$

Analogously

$$d_C(x) = \inf_{y \in C} \|x - y\| \geq \inf_{y \in C} (\|x_0 - y\| - \|x - x_0\|) =$$

$$= d_C(x_0) - \|x - x_0\| > 2\Delta - \Delta = \Delta.$$

Denoting $d := d_C(x)$, we have

$$\Delta < d < r - \Delta. \tag{1.10}$$

Take any $\varepsilon \in (0, \Delta)$. Let $a, b \in C$, $a \neq b$ be such that $a, b \in \varepsilon-\operatorname{argmin} d_C(x)$, and $\|a - b\| > \varepsilon$ (if any). Since $\|a - x\| \leq d + \varepsilon$ and $\|b - x\| \leq d + \varepsilon$ we have that

$$\|a - b\| \leq \|a - x\| + \|b - x\| \leq 2d + 2\varepsilon < 2(r - \Delta) + 2\Delta = 2r.$$

From (1.2) there exists $z \in C$ such that

$$\left\| \frac{a+b}{2} - z \right\| \leq r - \sqrt{r^2 - \frac{\|a - b\|^2}{4}}. \tag{1.11}$$

Setting

$$\overline{a} := x + d \frac{a - x}{\|a - x\|}$$

we have a point $\overline{a}$ such that

$$\|\overline{a} - x\| = \left\| d \frac{a - x}{\|a - x\|} \right\| = d$$

and since $\|a - x\| \le d + \varepsilon$

$$\begin{aligned} \|\overline{a} - a\| &= \Big\| x - a + d\frac{a-x}{\|a-x\|} \Big\| = \Big(1 - \frac{d}{\|a-x\|}\Big)\|a - x\| = \\ &= \|a - x\| - d \le d + \varepsilon - d = \varepsilon, \end{aligned}$$

i.e.

$$\|\overline{a} - x\| = d \text{ and } \|\overline{a} - a\| \le \varepsilon.$$

Analogously, setting

$$\overline{b} := x + d\frac{b-x}{\|b-x\|}$$

we obtain a point $\overline{b}$ such that

$$\|\overline{b} - x\| = d \text{ and } \|\overline{b} - b\| \le \varepsilon.$$

A sketch in $\mathbb{R}^2$ is given in Figure 1.5.

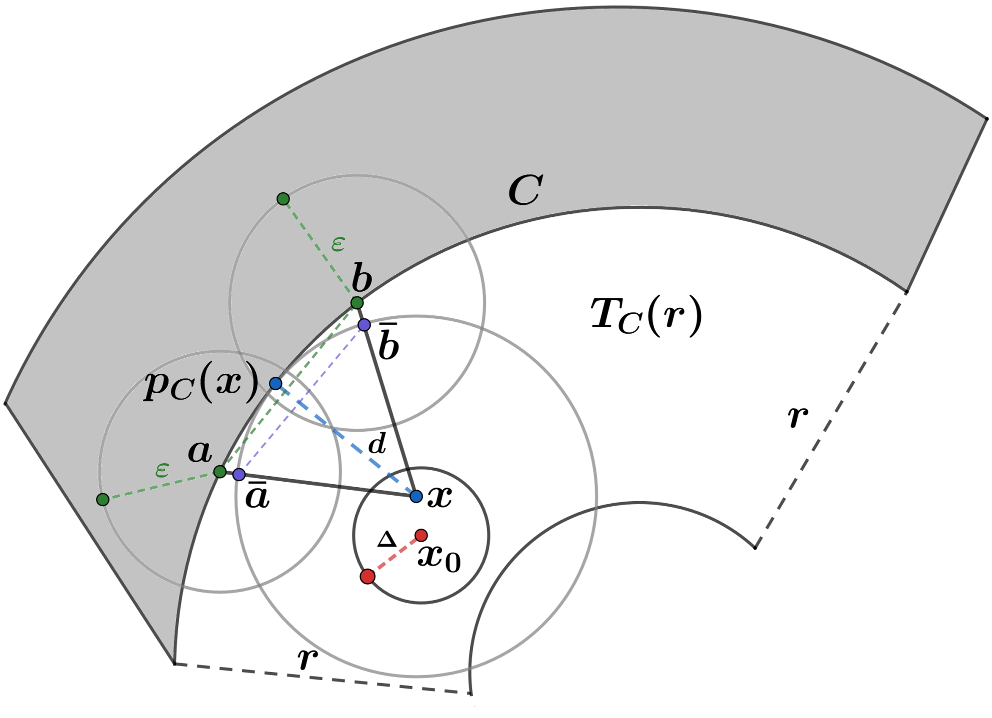

Figure 1.5: Geometric idea for the prove of $(c) \Rightarrow (a)$

Moreover, $\overline{a} \neq \overline{b}$ (otherwise one gets a contradiction with $\|a - b\| > \varepsilon$.)

We will show that

$$\|\overline{a}-\overline{b}\| \leq 2r\sqrt{\frac{2c(\Delta)}{\Delta}}\sqrt{\varepsilon}$$

and use it to give the upper bound

$$\|a-b\| \leq \left(2r\sqrt{\frac{2c(\Delta)}{\Delta}}+2\sqrt{\Delta}\right)\sqrt{\varepsilon}.$$

To this end from the parallelogram law we have that

$$2\left\|\frac{\overline{a}-x}{2}\right\|^2+2\left\|\frac{\overline{b}-x}{2}\right\|^2=\left\|\frac{\overline{a}+\overline{b}}{2}-x\right\|^2+\left\|\frac{\overline{a}-\overline{b}}{2}\right\|^2.$$

Hence

$$\begin{aligned}\left\|\frac{\overline{a}+\overline{b}}{2}-x\right\|^2 &= 2\left\|\frac{\overline{a}-x}{2}\right\|^2+2\left\|\frac{\overline{b}-x}{2}\right\|^2-\left\|\frac{\overline{a}-\overline{b}}{2}\right\|^2\\ &= \frac{1}{2}d^2+\frac{1}{2}d^2-\frac{\|\overline{a}-\overline{b}\|^2}{4}=d^2-\frac{\|\overline{a}-\overline{b}\|^2}{4},\end{aligned}$$

which yields that

$$\left\|\frac{\overline{a}+\overline{b}}{2}-x\right\|=\sqrt{d^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}. \tag{1.12}$$

Now we will show that any ball centered at $\dfrac{\overline{a}+\overline{b}}{2}$ with radius ρ_0 such than

$$0<\rho_0<d-\sqrt{d^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}$$

does not contain any point of the set C. Suppose that $z_0 \in C \cap B\left(\dfrac{\overline{a}+\overline{b}}{2},\rho_0\right)$.

From (1.12), the choice of ρ_0 and since $d=d_C(x)$ we get that

$$\begin{aligned} d &= d_C(x)=\inf_{y\in C}\|x-y\| \leq \|x-z_0\| \leq\\ &\leq \left\|\frac{\overline{a}+\overline{b}}{2}-x\right\|+\left\|\frac{\overline{a}+\overline{b}}{2}-z_0\right\| <\\ &< \sqrt{d^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}+\rho_0<d.\end{aligned}$$

which yields a contradiction.

But $z \in C$, hence it holds that

$$\left\| \frac{\bar{a}+\bar{b}}{2} - z \right\| \geq d - \sqrt{d^2 - \frac{1}{4}\|\bar{a}-\bar{b}\|^2}. \tag{1.13}$$

Combining (1.13) with (1.11) and using the triangle inequality we get

$$\begin{aligned} d - \sqrt{d^2 - \frac{1}{4}\|\bar{a}-\bar{b}\|^2} &\leq \left\| \frac{\bar{a}+\bar{b}}{2} - z \right\| = \left\| \frac{a+b}{2} - z + \left(\frac{\bar{a}+\bar{b}}{2} - \frac{a+b}{2} \right) \right\| \\ &\leq \left\| \frac{a+b}{2} - z \right\| + \frac{1}{2}\|\bar{a} - a + \bar{b} - b\| \\ &\leq \left\| \frac{a+b}{2} - z \right\| + \frac{1}{2}\left(\|a-\bar{a}\| + \|b-\bar{b}\| \right) \\ &\leq \left\| \frac{a+b}{2} - z \right\| + \varepsilon \\ &\leq r - \sqrt{r^2 - \frac{1}{4}\|a-b\|^2} + \varepsilon. \qquad (1.14) \end{aligned}$$

Again from the triangle inequality we have

$$\|a-b\| \leq \|a-\bar{a}\| + \|\bar{a}-\bar{b}\| + \|\bar{b}-b\| \leq \|\bar{a}-\bar{b}\| + 2\varepsilon,$$

which yields that

$$\frac{\|a-b\|^2}{4} \leq \frac{(\|\bar{a}-\bar{b}\| + 2\varepsilon)^2}{4}$$

and

$$r^2 - \frac{\|a-b\|^2}{4} \geq r^2 - \frac{(\|\bar{a}-\bar{b}\| + 2\varepsilon)^2}{4} > 0, \tag{1.15}$$

where the strict inequality in the latter holds since by (1.10) and $\varepsilon < \Delta$ we have

$$\|\bar{a}-\bar{b}\| + 2\varepsilon \leq \|\bar{a}-x\| + \|\bar{b}-x\| + 2\varepsilon = 2d + 2\varepsilon < 2(r-\Delta) + 2\Delta = 2r.$$

From (1.15) we get that

$$r - \sqrt{r^2 - \frac{\|a-b\|^2}{4}} \leq r - \sqrt{r^2 - \frac{(\|\bar{a}-\bar{b}\| + 2\varepsilon)^2}{4}}$$

which combined with (1.14) gives

$$d - \sqrt{d^2 - \frac{1}{4}\|\bar{a}-\bar{b}\|^2} \leq r - \sqrt{r^2 - \frac{1}{4}(\|\bar{a}-\bar{b}\| + 2\varepsilon)^2} + \varepsilon =$$

$$= r - \sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} + \sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} - \sqrt{r^2 - \frac{1}{4}(\|\overline{a} - \overline{b}\| + 2\varepsilon)^2} + \varepsilon.$$

To estimate the difference of the last two square roots we multiply and divide it by their sum to obtain

$$\sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} - \sqrt{r^2 - \frac{1}{4}(\|\overline{a} - \overline{b}\| + 2\varepsilon)^2} =$$

$$\frac{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2 - r^2 + \frac{1}{4}(\|\overline{a} - \overline{b}\| + 2\varepsilon)^2}{\sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} + \sqrt{r^2 - \frac{1}{4}(\|\overline{a} - \overline{b}\| + 2\varepsilon)^2}} =$$

$$= \frac{\|\overline{a} - \overline{b}\|\varepsilon + \varepsilon^2}{\sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} + \sqrt{r^2 - \frac{1}{4}(\|\overline{a} - \overline{b}\| + 2\varepsilon)^2}} \leq \frac{\|\overline{a} - \overline{b}\|\varepsilon + \varepsilon^2}{\sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2}}.$$

Since $\|\overline{a} - \overline{b}\| \leq \|\overline{a} - x\| + \|\overline{b} - x\| = d + d = 2d$, $\varepsilon < \delta$ and (1.10) we have that

$$\|\overline{a} - \overline{b}\|\varepsilon + \varepsilon^2 \leq 2d\varepsilon + \varepsilon^2 = \varepsilon(2d + \varepsilon) \leq \varepsilon(2(r - \Delta) + \Delta) < 2r\varepsilon,$$

and

$$\sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} \geq \sqrt{r^2 - d^2} = \sqrt{(r - d)(r + d)} \geq \sqrt{\Delta(r + \Delta)}. \geq \sqrt{r\Delta}$$

Hence we finally get the estimation

$$\sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} - \sqrt{r^2 - \frac{1}{4}(\|\overline{a} - \overline{b}\| + 2\varepsilon)^2} \leq \left(2\sqrt{\frac{r}{\Delta}}\right)\varepsilon,$$

which yields

$$d - \sqrt{d^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} \leq r - \sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} + \left(2\sqrt{\frac{r}{\Delta}} + 1\right)\varepsilon,$$

and setting $c(\Delta) := \left(2\sqrt{\frac{r}{\Delta}} + 1\right)$ we have that

$$d - \sqrt{d^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} \leq r - \sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} + c(\Delta)\varepsilon. \tag{1.16}$$

Now for a fixed $t \geq 0$ we consider the function

$$f : [\sqrt{t}, \infty) \to \mathbb{R}^+$$

defined as

$$f(r) := r - \sqrt{r^2 - t}.$$

It is a convex decreasing function whose derivative at $r > \sqrt{t}$ is

$$f'(r) = 1 - \frac{r}{\sqrt{r^2 - t}}.$$

Moreover, for $r \geq \sqrt{t}$ we have that

$$f(r) \geq \frac{t}{2r}. \tag{1.17}$$

The latter is proved in Proposition A.1.4 in the Appendix.

Taking $t := \frac{1}{4}\|\bar{a} - \bar{b}\|^2$, in the definition of f, the inequality (1.16) can be written as

$$f(d) \leq f(r) + c(\Delta)\varepsilon,$$

or

$$f(d) - f(r) \leq c(\Delta)\varepsilon.$$

Note that since $\|a - b\| < 2r$ we have that $r \geq \sqrt{t}$ which assures that this choice of t is compatible with the function f.

The convexity and differentiability of f yield that

$$f'(r)(d - r) \leq f(d) - f(r) \leq c(\Delta)\varepsilon.$$

The latter reads

$$\left(1 - \frac{r}{\sqrt{r^2 - t}}\right)(d - r) \leq c(\Delta)\varepsilon.$$

Hence

$$(r - d)\left(r - \sqrt{r^2 - \frac{\|\bar{a} - \bar{b}\|^2}{4}}\right) \leq c(\Delta)\varepsilon\sqrt{r^2 - \frac{\|\bar{a} - \bar{b}\|^2}{4}}. \tag{1.18}$$

Using that

$$\left(r - \sqrt{r^2 - \frac{\|\bar{a} - \bar{b}\|^2}{4}}\right) \geq \frac{\|\bar{a} - \bar{b}\|^2}{8r},$$

see (1.17) and that

$$\sqrt{r^2 - \frac{\|\bar{a} - \bar{b}\|^2}{4}} \leq r,$$

from (1.18) we obtain

$$(r - d)\frac{\|\bar{a} - \bar{b}\|^2}{8r} \leq rc(\Delta)\varepsilon.$$

As $r - d > \Delta$, see (1.10),

$$\|\overline{a} - \overline{b}\|^2 \leq \frac{8r^2}{\Delta} c(\Delta)\varepsilon,$$

hence

$$\|\overline{a} - \overline{b}\| \leq 2r\sqrt{\frac{2c(\Delta)}{\Delta}}\sqrt{\varepsilon}.$$

From the latter and $\varepsilon < \Delta$ we get

$$\|a - b\| \leq \|\overline{a} - \overline{b}\| + 2\varepsilon \leq \left(2r\sqrt{\frac{2c(\Delta)}{\Delta}} + 2\sqrt{\varepsilon}\right)\sqrt{\varepsilon} \leq \left(2r\sqrt{\frac{2c(\Delta)}{\Delta}} + 2\sqrt{\Delta}\right)\sqrt{\varepsilon}.$$

And finally denoting

$$k = k(\Delta) := 2r\sqrt{\frac{2c(\Delta)}{\Delta}} + 2\sqrt{\Delta},$$

we have that for $a, b \in \varepsilon{-}\operatorname{argmin} d_C(x)$ such that $\|a - b\| > \varepsilon$ we have

$$\|a - b\| \leq k\sqrt{\varepsilon}.$$

If we consider $a, b \in \varepsilon{-}\operatorname{argmin} d_C(x)$ such that $\|a - b\| \leq \varepsilon$, then since clearly $k > \sqrt{\varepsilon}$, we will also have $\|a - b\| \leq k\sqrt{\varepsilon}$.

Therefore,

$$\operatorname{diam}\left(\varepsilon{-}\operatorname{argmin} d_C(x)\right) \leq k\sqrt{\varepsilon}. \tag{1.19}$$

This means that the projection mapping is single-valued on $B(x_0, \Delta)$, i.e. for $x \in B(x_0, \Delta)$ there exists unique point $p_C(x) \in C$ such that

$$d_C(x) = \|x - p_C(x)\|.$$

As $x_0 \in T_C(r)$ was arbitrary, the projection mapping P_C is single-valued on $T_C(r)$.

It is routine to establish the continuity of the metric projection mapping P_C at x_0.

Take $x, y \in B(x_0, \Delta/4)$. For their projections we have that $\|x - p_C(x)\| = d_C(x)$ and $\|y - p_C(y)\| = d_C(y)$.

Since the distance function d_C is Lipschitz continuous with constant 1 (see Proposition A.1.5 in the Appendix), i.e.

$$|d_C(x) - d_C(y)| \leq \|x - y\|, \text{ for all } x, y \in H,$$

we have that

$$\begin{aligned} \|p_C(y) - x\| &\leq \|p_C(y) - y\| + \|y - x\| \\ &= d_C(y) + \|y - x\| \\ &\leq d_C(x) + 2\|y - x\|. \end{aligned}$$

Hence we get

$$p_C(y) \in (2\|y-x\|) - \operatorname{argmin} d_C(x).$$

Obviously $p_C(x) \in (2\|y-x\|) - \operatorname{argmin} d_C(x)$.

Note that for $\varepsilon := 2\|y-x\|$ we have that

$$\varepsilon = 2\|y-x\| \leq 2\left(\|y-x_0\| + \|x-x_0\|\right) < 2\left(\frac{\Delta}{4} + \frac{\Delta}{4}\right) = \Delta.$$

And finally from (1.19) we get that

$$\|p_C(y) - p_C(x)\| \leq \sqrt{2}k\sqrt{\|y-x\|}.$$

The latter yields that P_C is norm-to-norm continuous at x_0, and as x_0 was arbitrary in $T_C(r)$, on $T_C(r)$. From Theorem 1.1.3 (c) it holds that C is r-prox-regular, thus completing the proof of $(c) \Rightarrow (a)$. □

From the proof of Theorem 1.1.1 it is clear that the property: the projection mapping P_C is single-valued and norm-to-norm continuous on $T_C(r)$ also characterizes r-prox-regular closed set C, but it is an external characterization.

Chapter 2

Epigraphical characterization of uniformly lower regular functions

The concept of a primal lower nice function was introduced by Poliquin in [41] where it was proved that Clarke and proximal subdifferentials of a primal lower nice function defined on finite-dimensional space coincide. In particular this means that if the definition of primal lower nice property, see (2.5), is taken with respect to the Clarke subdifferential, this will produce the same class of functions. In [41] Poliquin proved that these functions in $\mathbb{R}^n$ are completely characterized by their Clarke subdifferential. This was the first large class of non-convex lower semicontinuous functions with this property.

The coincidence of proximal and Clarke subdifferentials of a primal lower nice function defined on Hilbert space was proved by Levy, Poliquin and Thibault in [35]. Later Ivanov and Zlateva in [28] showed that Clarke and proximal subdifferential of a primal lower nice function defined on a β smooth Banach space coincide. The result obtained in [28] shows that the class of primal lower nice functions does not depend on what reasonable subdifferential is used in defining the class. Ivanov and Zlateva in [28] suggested "that it is possible to characterize primal lower nice property in terms not involving subdifferentials". As a step in this direction we prove that primal lower nice functions on Hilbert space satisfy a property which does not involve subdifferentials, see Theorem 2.4.1(i) and Corollary 2.4.2.

Since the pioneering work of Poliquin [41], primal lower nice functions are studied in a series of publications, see e.g. [42, 43, 27, 9, 52, 36]. These functions are closely related to prox-regular sets, a term due to Poliquin, Rockafellar and Thibault [44]. Indeed, a set in a Hilbert space is prox-regular exactly when its indicator function is primal lower nice, see [44, Proposition 2.1]. The study of prox-regular sets can be traced back to the pioneering work of Federer [23] who introduced them as positively reached sets in $\mathbb{R}^n$. During the years, various names of such sets have been introduced: weakly convex [53] or proximally smooth sets [19] are commonly used in Hilbert spaces; for other names see the survey [20]. Prox-regular sets in Banach spaces are studied in [10, 11, 24, 26, 6, 8, 33] and many others.

Along with the study of prox-regular sets from a theoretical point of view, they are intensively studied and involved in the famous Moreau's sweeping processes, see e.g. [2], the survey [39] and the references therein. Various properties of prox-regular sets are established in [1, 3, 4, 5]. More details one can find in the paper [44], the survey [20], the forthcoming book of Thibault [51], as well as, the bibliography therein.

Prox-regularity has been introduced as an important new regularity property in Variational Analysis by Poliquin and Rockafellar in [43], see also Chapter 13F in the monograph of Rockafellar and Wets [47]. They defined the concept for functions and sets and developed the subject in $\mathbb{R}^n$. Numerous significant characterizations of prox-regularity of a closed set C in Hilbert space at point $\overline{x} \in C$ were obtained by Poliquin, Rockafellar and Thibault in [44] in terms of the distance function d_C and metric projection mapping P_C, e.g. d_C being continuously differentiable outside of C on a neighbourhood of $\overline{x}$, or P_C being single-valued and norm-to-weak continuous on this same neighbourhood. On global level, in [44] the authors showed that uniformly prox-regular sets are proximally smooth sets providing new insights on them.

To prove our main result, we introduce and study the epi uniform prox-regularity of an epigraph set, see Definition 2.1.3. This notion slightly differs from the usual uniform prox-regularity of an arbitrary set, see Definition 2.1.1. We choose to work on global level, i.e. with uniform properties of functions and sets involved, but similar results easily can be obtained at local level as well. The properties of epi uniformly prox-regular sets in Hilbert space are also studied, see Section 2.3. Our main result is Theorem 2.4.1 where we prove the epigraphical characterization of uniformly lower regular functions on a Hilbert space – a class of functions containing uniformly primal lower nice functions. It reveals their distant resemblance to convex functions, see Corollary 2.4.2.

In the following Section 2.1 we give some notations, definitions and necessary preliminaries. In Section 2.2 it is proved that uniformly lower regular functions are exactly those with epi uniformly prox-regular epigraphs, see Theorem 2.2.1 and Theorem 2.2.2. In Section 2.3 are established some basic and important properties of epi uniformly prox-regular sets in $H \times \mathbb{R}$. The proof of our main result, Theorem 2.4.1, is given in the final Section 2.4.

The results in this chapter are published in [34].

2.1 Preliminaries

Together with the proximal normal come N_C we will use also the *Fréchet normal cone* $N_C^F(x)$ of C at x which consists of all $x^* \in H$ such that for any $\varepsilon > 0$ there exists a neighbourhood U of x such that the inequality $\langle x^*, x' - x\rangle \leq \varepsilon \|x' - x\|$ holds for all $x' \in C \cap U$.

It is not difficult to achieve that (see Proposition A.1.6 in the Appendix.)

$$N_C(x) \subseteq N_C^F(x), \quad \forall x \in C. \tag{2.1}$$

The definition of an uniformly prox-regular set in H is well-known, see e.g. [44, 10, 11]. A nonempty closed subset C of H is *uniformly prox-regular* if there is some $r > 0$ such that for any $x \in C$ and $p \in N_C(x) \cap \mathbb{B}$ one has

$$\langle p, x' - x\rangle \leq \frac{1}{2r}\|x' - x\|^2, \quad \forall x' \in C. \tag{2.2}$$

It is not difficult to see that it is equivalent to the following

Definition 2.1.1. *A nonempty closed subset C of H is* uniformly prox-regular *if there is $r > 0$ such that for any $x \in C$ and $p \in N_C(x) \cap \mathbb{B}_H$ one has*

$$\langle p, x' - x\rangle \leq \frac{1}{2r}\|x' - x\|^2, \quad \forall x' \in B(x, 2r) \cap C. \tag{2.3}$$

Indeed, if C is uniformly prox-regular according to Definition 2.1.1 then (2.2) holds for some $r > 0$ and any $x \in C$, $p \in N_C(x) \cap \mathbb{B}_H$.

If $x' \in C$ is such that $\|x' - x\| \geq 2r$, then

$$\langle p, x' - x\rangle \leq \|x' - x\| = \frac{\|x' - x\|^2}{\|x' - x\|} \leq \frac{1}{2r}\|x' - x\|^2,$$

so (2.2) holds.

If a set $C \subset H$ satisfies Definition 2.1.1, we will say that C is r prox-regular (omitting "uniformly" for brevity).

We will consider the space

$$\overline{H} := H \times \mathbb{R}$$

with the norm

$$|||(x, r)||| := \sqrt{\|x\|^2 + r^2},$$

for $(x, r) \in \overline{H}$. Then $(\overline{H}, |||\cdot|||)$ is a Hilbert space.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function. The *domain* of f is the set

$$\operatorname{dom} f := \{x \in H : f(x) \in \mathbb{R}\}$$

and the *epigraph* of f is the set

$$\operatorname{epi} f := \{(x, r) \in \overline{H} : r \geq f(x)\}.$$

The function f is *proper* exactly when $\operatorname{dom} f \neq \varnothing$ and f is lower semicontinuous on H exactly when $\operatorname{epi} f$ is closed in $\overline{H}$.

The *proximal subdifferential* of f at $x \in \operatorname{dom} f$ is defined as the set

$$\partial^p f(x) := \{p \in H | (p, -1) \text{ is a proximal normal to } \operatorname{epi} f \text{ at } \big(x, f(x)\big)\},$$

while $\partial^p f(x) = \varnothing$ for $x \notin \operatorname{dom} f$, see e.g. [11, p. 2216].

Thus, by definition,

$$p \in \partial^p f(x) \iff (p, -1) \in N_{\operatorname{epi} f}\big(x, f(x)\big). \tag{2.4}$$

Since the concept for a primal lower nice function at a point of its domain was introduced by Poliquin [41], such functions defined on Hilbert space have been extensively studied ever since, see e.g. [21, 41, 42, 35, 52]. In [10, 11] for a function defined on a uniformly convex Banach space the J primal lower regular (J-plr in short) concept at a point of its domain was introduced, where J stands for the duality mapping. In [30, 31] for a function on a Banach space was studied the s-lower regular concept. For a function on a Hilbert space both J-plr and 1-lower regular concept at a point of its domain coincide with the primal lower nice one.

When the constants involved in the definition of the primal lower nice property are uniform, one speaks about uniform lower nice property.

A proper lower semicontinuous function $f : H \to \mathbb{R}$ is said to be *uniformly primal lower nice* if there exist $\rho > 0$ and $\theta > 0$ such that for any $t \geq \theta$, any $p \in \partial^p f(x)$ with $\|p\| \leq \rho t$,

$$f(x') \geq f(x) + \langle p, x' - x\rangle - \frac{t}{2}\|x' - x\|^2, \quad \text{for all } x' \in H, \tag{2.5}$$

see [11, p. 2226].

From the very definition, it is clear that if f is uniformly primal lower nice with some positive constants ρ, and θ, then it is so for any $\rho' < \rho$ and $\theta' > \theta$. Hence, taking small ρ, and then $\theta = \rho^{-1}$ one comes to the following equivalent definition: a proper lower semicontinuous function $f : H \to \mathbb{R}$ is uniformly primal lower nice if there exists $\rho > 0$ such that for any $t \geq \rho^{-1}$, and any $p \in \partial f(x)$ with $\|p\| \leq \rho t$, (2.5) holds. When the latter holds for f for some $\rho > 0$ one says that the function f is ρ primal lower nice (omitting "uniformly" for brevity).

It is easy to see that such functions are, for example, the 1-lower regular on the whole space H functions [30, 31].

Further we will consider a slightly more general definition for uniform epi lower regularity of a function.

Definition 2.1.2. *A proper lower semicontinuous function $f : H \to \mathbb{R}\cup\{+\infty\}$ is said to be epi uniformly lower regular if there exists $\rho > 0$ such that for any $t \geq \rho^{-1}$, any $p \in \partial^p f(x)$ with $\|p\| \leq \rho t$, it is true that*

$$\alpha' \geq f(x) + \langle p, x' - x\rangle - \frac{t}{2}\|x' - x\|^2, \tag{2.6}$$

$$\textit{for all } (x', \alpha') \in B((x, f(x)), 2\rho) \cap \operatorname{epi} f.$$

If a function f satisfies Definition 2.1.2 for some $\rho > 0$, we will say that f is epi ρ lower regular (again omitting "uniformly"). It is clear that any ρ primal lower-nice function is epi ρ lower regular.

A non-empty closed set $C \subset \overline{H}$ will be called an *epigraph set* if $C \equiv \operatorname{epi} f$ for a proper lower semicontinuous function $f : H \to \mathbb{R} \cup \{+\infty\}$.

For an epigraph set in $\overline{H}$ we will introduce the notion of epi uniform prox-regularity which slightly differs from well-known uniform prox-regularity of a set in $\overline{H}$.

Definition 2.1.3. *Let C be an epigraph set in $\overline{H}$. One says that C is uniformly epi prox-regular if there is $r > 0$ such that for any $(x, \alpha) \in C$, and $(q, \eta) \in N_C\big(x, \alpha\big) \cap \mathbb{B}_{\overline{H}}$ one has*

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|x' - x\|^2, \tag{2.7}$$

$$\textit{for all } (x', \alpha') \in B((x, \alpha), 2r) \cap C.$$

If an epigraph set C satisfies Definition 2.1.3 for some $r > 0$, we will say that C is epi r prox-regular (omitting "uniformly").

From the very definitions it is clear that if an epigraph set $C \subset \overline{H}$ is epi r prox-regular according to Definition 2.1.3, then C is r prox-regular set in $\overline{H}$ according to Definition 2.1.1.

Indeed, if $(x, \alpha) \in C$, and $(q, \eta) \in N_C\big(x, \alpha\big) \cap \mathbb{B}_{\overline{H}}$ from (2.7) it follows that for all $(x', \alpha') \in B((x, \alpha), 2r) \cap C$,

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|x' - x\|^2,$$

so

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|x, - x\|^2 \leq \frac{1}{2r} |||(x' - x, \alpha' - \alpha)|||^2,$$

which is (2.3) in $\overline{H} = H \times \mathbb{R}$.

From uniform prox-regularity of an epigraph set it does not hold in general that it is an epi uniformly prox-regular set. Before proceeding with the rest of this chapter, let us note that the uniform results obtained in the rest of this chapter have their local counterparts proven in the same manner.

2.2 Epi uniformly lower regular functions

First we will prove that if $f : H \to \mathbb{R} \cup \{+\infty\}$ is an epi uniformly lower regular function, then $\operatorname{epi} f$ is an epi uniformly prox-regular set in $\overline{H}$. The proof follows the lines of the proofs of [11, Propositions 4.1 and 4.4] where J-plr functions are considered.

Theorem 2.2.1. *If $f : H \to \mathbb{R} \cup \{+\infty\}$ is epi ρ lower regular function, then $C \equiv \operatorname{epi} f$ is epi ρ prox-regular set in $\overline{H}$.*

Proof. Let $(x, \alpha) \in C$ and $(x^*, -\lambda) \in N_C(x, \alpha) \cap \mathbb{B}_{\overline{H}}$. It is routine to show that since $(x^*, -\lambda) \in N_C(x, \alpha) \cap \mathbb{B}_{\overline{H}}$ we have that $-\lambda \leq 0$, or equivalently $\lambda \geq 0$ (see Proposition A.1.7 in the Appendix). So will consider the following two cases:

Case 1. $\lambda > 0$. In this case we have that $f(x) = \alpha$ (see Proposition A.1.8 in the Appendix). Since $N_C\big(x, f(x)\big)$ is a cone (see Corollary A.1.2 in the Appendix)

$$\big(\lambda^{-1}x^*, -1\big) \in N_C\big(x, f(x)\big).$$

From (2.4) it holds that

$$\lambda^{-1}x^* \in \partial^p f(x).$$

Since $(x^*, -\lambda) \in \mathbb{B}_{\overline{H}}$ we have that $\|x^*\| \leq 1$, hence

$$\big\|\lambda^{-1}x^*\big\| \leq \frac{1}{\lambda}.$$

Let us take $t = \frac{1}{\lambda\rho}$. So,

$$\big\|\lambda^{-1}x^*\big\| \leq t\rho.$$

Since f is epi ρ lower regular function, $\lambda^{-1}x^* \in \partial^p f(x)$, and $\|\lambda^{-1}x^*\| \leq t\rho$ we get that for all $(x', \alpha') \in B\left(\big(x, f(x)\big), 2\rho\right) \cap \operatorname{epi} f$,

$$\alpha' \geq f(x) + \langle \lambda^{-1}x^*, x' - x\rangle - \frac{t}{2}\|x' - x\|^2,$$

Multiplying by $\lambda > 0$ in the latter and using that $f(x) = \alpha$ we obtain,

$$0 \geq \lambda(\alpha - \alpha') + \langle x^*, x' - x\rangle - \frac{\lambda t}{2}\|x' - x\|^2,$$

and by the choice of t we get

$$\langle (x^*, -\lambda), (x' - x, \alpha' - \alpha)\rangle \leq \frac{1}{2\rho}\|x' - x\|^2, \quad \forall\ (x', \alpha') \in B\left(\big(x, f(x)\big), 2\rho\right) \cap C.$$

Which means that for $\lambda > 0$ the set C is epi ρ prox-regular in $\overline{H}$. The proof of this case is complete.

Case 2. $\lambda = 0$. In this case we have that $(x^*, 0) \in N_C(x, \alpha)$. From [11, Lemma 4.2] it holds that

$$(x^*, 0) \in N_C\big(x, f(x)\big),$$

hence by inclusion (2.1) we achieve

$$(x^*, 0) \in N_C^F\big(x, f(x)\big).$$

Using the approximation result of Ioffe [25, p. 190], we can find sequences

$$\{\lambda_n\},\ \{u_n\},\ \{u_n^*\}$$

such that

$$\begin{gathered}\lambda_n \searrow 0,\ \text{ as } n \to \infty,\\ (u_n^*, -\lambda_n) \in N_C^F\big(u_n, f(u_n)\big),\ \text{ for all } n \in \mathbb{N},\\ \big(u_n, f(u_n)\big) \to \big(x, f(x)\big),\ \text{ as } n \to \infty,\end{gathered} \tag{2.8}$$

and

$$|||(u_n^*, -\lambda_n) - (x^*, 0)||| \to 0 \text{ as } n \to \infty. \tag{2.9}$$

Further, we use the approximation result in [11, Proposition 3.1] to find sequences

$$\big\{(x_n, \alpha_n)\big\} \text{ and } \big\{(y_n^*, -\mu_n)\big\}$$

such that

$$(x_n, \alpha_n) \in C \ \text{ for all } n \in \mathbb{N},$$

$$(y_n^*, -\mu_n) \in N_C\big(x_n, \alpha_n\big),\ \text{ for all } n \in \mathbb{N}, \tag{2.10}$$

$$|||(x_n, \alpha_n) - \big(u_n, f(u_n)\big)||| < \frac{\lambda_n}{2},\ \text{ for all } n \in \mathbb{N}, \tag{2.11}$$

and

$$|||(y_n^*, -\mu_n) - (u_n^*, -\lambda_n)||| < \frac{\lambda_n}{2},\ \text{ for all } n \in \mathbb{N}, \tag{2.12}$$

From (2.12) it follows that $|\lambda_n - \mu_n| < \frac{\lambda_n}{2}$, hence $\frac{\lambda_n}{2} < \mu_n < \frac{3\lambda_n}{2}$, so $\mu_n \searrow 0$. Let us denote

$$x_n^* := \mu_n^{-1} y_n^*.$$

From (2.10) and since $\mu_n > 0$ we have that

$$f(x_n) = \alpha_n \ \text{ for all } n \in \mathbb{N}$$

(see Proposition A.1.8 in the Appendix). The latter yields that

$$(y_n^*, -\mu_n) \in N_C\big(x_n, f(x_n)\big),\ \text{ for all } n \in \mathbb{N}.$$

Hence

$$(x_n^*, -1) \in N_C\big(x_n, f(x_n)\big)$$

and therefore, $x_n^* \in \partial^p f(x_n)$, (see (2.4)). We will show that

$$x_n \to x, \quad f(x_n) \to f(x), \quad \mu_n x_n^* \to x^*, \text{ as } n \to \infty. \tag{2.13}$$

Form (2.11) we have that

$$\|x_n - u_n\| \le |||(x_n, \alpha_n) - \big(u_n, f(u_n)\big)||| < \frac{\lambda_n}{2}$$

hence $\|x_n - u_n\|$ tends to zero as $n \to \infty$. Analogously form (2.2) $\|u_n - x\|$ tends to zero as $n \to \infty$. Using the inequality we have that

$$\|x_n - x\| \le \|x_n - u_n\| + \|u_n - x\|$$

therefore $x_n \to x$, as $n \to \infty$. From $f(x_n) = \alpha_n$ and the triangle inequality we have

$$|f(x_n) - f(x)| = |\alpha_n - f(x)| \le |\alpha_n - f(u_n)| + |f(u_n) - f(x)|,$$

and because of (2.11) and (2.2) we can achieve that $f(x_n) \to f(x)$, as $n \to \infty$. As $\mu_n x_n^* = y_n^*$ again form the triangle inequality one has

$$\|\mu_n x_n^* - x^*\| = \|y_n^* - x^*\| \le \|y_n^* - u_n^*\| + \|u_n^* - x^*\|,$$

hence $\mu_n x_n^* \to x^*$ can be shown using (2.12) and (2.9). Assume for a while that $x^* \neq 0$ and let us denote

$$t_n := \max\left(\frac{1}{\rho\mu_n}, \frac{\|x_n^*\|}{\rho\|x^*\|}\right).$$

Obviously when n goes to infinity,

$$\mu_n t_n = \max\left(\frac{1}{\rho}, \frac{\mu_n\|x_n^*\|}{\rho\|x^*\|}\right) \to \frac{1}{\rho}. \tag{2.14}$$

Now let $(x', \alpha') \in B\left(\big(x, f(x)\big), 2\rho\right) \cap \operatorname{epi} f$ be arbitrary. For a sufficiently large n we would have that

$$(x', \alpha') \in B\left(\big(x_n, f(x_n)\big), 2\rho\right) \cap \operatorname{epi} f \text{ and } t_n \ge \frac{1}{\rho}$$

Hence since f is an epi ρ lower regular function and $x_n^* \in \partial^p f(x_n)$ with $\|x_n^*\| \le t_n\rho$, we have that

$$\alpha' \ge f(x_n) + \langle x_n^*, x' - x_n\rangle - \frac{t_n}{2}\|x' - x_n\|^2.$$

Multiplying by $\mu_n > 0$ in the latter we get

$$0 \geq \mu_n(f(x_n) - \alpha') + \langle \mu_n x_n^*, x' - x_n \rangle - \frac{\mu_n t_n}{2} \|x' - x_n\|^2.$$

Now letting n tend to infinity and using (2.13), and (3.16) we obtain

$$0 \geq \langle x^*, x' - x \rangle - \frac{1}{2\rho} \|x' - x\|^2.$$

Since the latter obviously holds for $x^* = 0$, the proof in this case and hence as a whole is complete. □

Now we will prove the converse, i.e. that if $C \equiv \operatorname{epi} f$ is an epi uniformly prox-regular set in $\overline{H}$, then f is an epi uniformly lower regular function on H.

Theorem 2.2.2. *If the epigraph set $C \equiv \operatorname{epi} f$ in $\overline{H}$ is epi r prox-regular, then the corresponding $f : H \to \mathbb{R} \cup \{+\infty\}$ is epi ρ lower regular function for*

$$\rho = \frac{r}{\sqrt{2}}.$$

Proof. Let $p \in \partial^p f(x)$ be such that $\|p\| \leq \rho t$ for some $t \geq \rho^{-1}$.

From (2.4) we have that

$$(p, -1) \in N_C\big(x, f(x)\big),$$

hence

$$\frac{1}{\sqrt{\|p\|^2 + 1}} (p, -1) \in N_C\big(x, f(x)\big) \cap \mathbb{B}_{\overline{H}}.$$

As the set C is epi r prox-regular and $\rho < r$, for all $(x', \alpha') \in B((x, f(x)), 2\rho) \cap C$,

$$\frac{1}{\sqrt{\|p\|^2 + 1}} \big\langle (p, -1), \big(x' - x, \alpha' - f(x)\big) \big\rangle \leq \frac{1}{2r} \|x' - x\|^2,$$

hence,

$$\langle p, x' - x \rangle + f(x) - \alpha' \leq \frac{\sqrt{\|p\|^2 + 1}}{2r} \|x' - x\|^2.$$

Therefore,

$$\alpha' \geq f(x) + \langle p, x' - x \rangle - \frac{\sqrt{\|p\|^2 + 1}}{2r} \|x' - x\|^2. \tag{2.15}$$

Using that $\|p\| \leq t\rho$, $\dfrac{1}{t} \leq \rho$ and $\rho = \dfrac{r}{\sqrt{2}}$, we get

$$\|p\|^2 + 1 \leq t^2\rho^2 + 1 = t^2\left(\rho^2 + \frac{1}{t^2}\right) \leq 2t^2\rho^2 = r^2t^2,$$

hence,

$$-\frac{\sqrt{\|p\|^2+1}}{r} \geq -\frac{tr}{r} = -t. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$\alpha' \geq f(x) + \langle p, x' - x\rangle - \frac{t}{2}\|x' - x\|^2, \ \forall (x', \alpha') \in B((x, f(x)), 2\rho) \cap C,$$

which means that f is epi ρ lower regular. □

It is worth to note here that using the same technique one can prove that f is ρ primal lower-nice function exactly when $C \equiv \operatorname{epi} f$ is such that for any $(x, \alpha) \in C$, and $(q, \eta) \in N_C\big(x, \alpha\big)$ with $q \in B_H$ one has

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha)\big\rangle \leq \frac{1}{2r}\|x' - x\|^2, \quad \forall (x', \alpha') \in B((x, \alpha), 2r) \cap C.$$

2.3 Properties of epi uniformly prox-regular sets

Recall that in Chapter 1 we examined the well-known characteristic property for a r prox-regular set set C in a Hilbert space H: for any $a, b \in C$ with $\|a - b\| < 2r$ and any $\lambda \in (0, 1)$ for

$$x_\lambda := \lambda a + (1 - \lambda)b$$

there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \leq \varphi(\lambda),$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1 - \lambda)\|a - b\|^2}.$$

Now we will use arguments in the line of the proof of Theorem 1.1.1) to show that epi uniformly prox regular set in $\overline{H}$ possesses the following similar property as well.

Theorem 2.3.1. *Let $C \subset \overline{H}$ be an epi r prox-regular set in $\overline{H}$. Let $(a, \alpha), (b, \beta) \in C$ be such that*

$$|||(a, \alpha) - (b, \beta)||| < 2r.$$

Then for any $\lambda \in [0,1]$ *and* $(x_\lambda, \gamma_\lambda)$*, where*

$$x_\lambda := \lambda a + (1-\lambda)b \text{ and } \gamma_\lambda := \lambda\alpha + (1-\lambda)\beta$$

there exists $(u_\lambda, \xi_\lambda) \in C$ *such that*

$$d_C(x_\lambda, \gamma_\lambda) = |||(x_\lambda, \gamma_\lambda) - (u_\lambda, \xi_\lambda)||| \leq \varphi(\lambda), \tag{2.17}$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Note that $\varphi(\lambda)$ doesn't depend on α and β, i.e. thanks to Theorem 2.3.1 if $|||(a,\alpha) - (b,\beta)||| < 2r$ we can give a upper estimate of $d_C(x_\lambda, \gamma_\lambda)$ using only λ, r and $\|a-b\|$. In Figure 2.1 is given a sketch in $\mathbb{R}^2$.

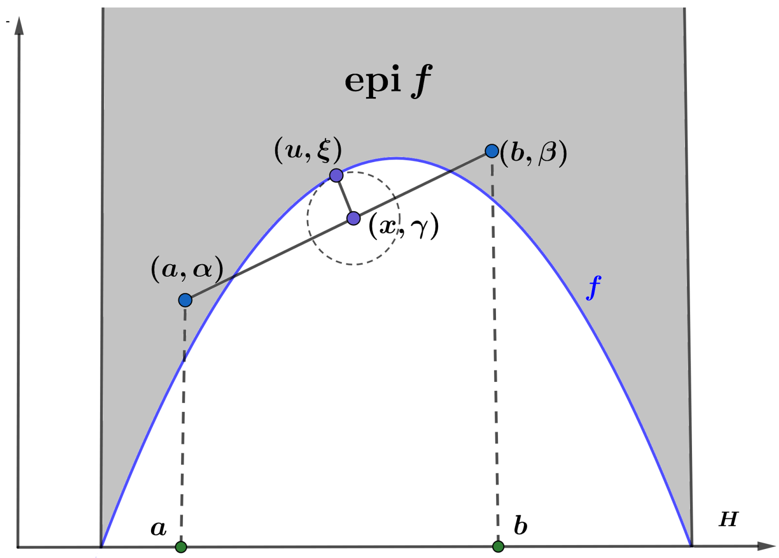

Figure 2.1: Geometric interpretation of Theorem 2.3.1 in $\mathbb{R}^2$

Proof. Take an arbitrary $\lambda \in [0,1]$ and consider the corresponding to it $(x_\lambda, \gamma_\lambda)$. We fix λ and further we will omit it from the index. First we will show that $(x,\gamma) \in U_C(r)$. Since $(a,\alpha), (b,\beta) \in C$ by the definition of the distance function we have that

$$d_C(x,\gamma) \leq |||(x,\gamma) - (a,\alpha)||| = (1-\lambda)|||(a,\alpha) - (b,\beta)|||$$

and

$$d_C(x,\gamma) \leq |||(x,\gamma) - (b,\beta)||| = \lambda|||(a,\alpha) - (b,\beta)|||.$$

Hance we get that

$$d_C(x,\gamma) \le \min(1-\lambda,\lambda)\,|||(a,\alpha)-(b,\beta)||| \le \frac{1}{2}|||(a,\alpha)-(b,\beta)||| < r,$$

which means that $(x,\gamma) \in U_C(r)$. If $(x,\gamma) \in C$ then (2.17) holds for $(u,\xi) = (x,\gamma)$. Now, consider the case $(x,\gamma) \notin C$. Since $(x,\gamma) \in T_C(r)$ and the set C is prox-regular, there exist a unique $(u,\xi) \in C$ such that

$$(u,\xi) = p_C(x,\gamma).$$

Denote

$$(p,\eta) := (x-u,\gamma-\xi).$$

So, $|||(p,\eta)||| \neq 0$, $\|p\| \le |||(p,\eta)||| < r$ and

$$u = \lambda a + (1-\lambda)b - p, \quad \xi = \gamma - \eta. \tag{2.18}$$

As $(u,\xi) \in P_C(x,\gamma)$, it holds that $0 \neq (p,\eta) \in N_C(u,\xi)$. Hence, $\eta \le 0$ or, equivalently, $\gamma \le \xi$ (see Proposition A.1.7 in the Appendix). Since $\dfrac{(p,\eta)}{|||p,\eta)|||} \in N_C(u,\xi) \cap \mathbb{B}_{\overline{H}}$ and C is epi r prox-regular set, we have that for all $(x',\alpha') \in C$ such that $|||(x',\alpha')-(u,\xi)||| < 2r$ it holds that

$$\frac{1}{|||(p,\eta)|||}\langle (p,\eta),(x',\alpha')-(u,\xi)\rangle \le \frac{1}{2r}\|x'-u\|^2. \tag{2.19}$$

Since

$$\begin{aligned}
|||(a,\alpha)-(u,\xi)||| &\le |||(a,\alpha)-(x,\gamma)||| + |||(x,\gamma)-(u,\xi)||| = \\
&= (1-\lambda)|||(a,\alpha)-(b,\beta)||| + |||(x,\gamma)-(u,\xi)||| \le \\
&\le (1-\lambda)|||(a,\alpha)-(b,\beta)||| + |||(x,\gamma)-(b,\beta)||| = \\
&= (1-\lambda)|||(a,\alpha)-(b,\beta)||| + \lambda|||(a,\alpha)-(b,\beta)||| = \\
&= |||(a,\alpha)-(b,\beta)||| < 2r,
\end{aligned}$$

we can put $(x',\alpha') = (a,\alpha)$ in (2.19) to get

$$\langle p, a-u\rangle + \eta(\alpha-\xi) \le \frac{|||(p,\eta)|||}{2r}\|a-u\|^2.$$

Using the expressions for u and ξ from (2.18) in the latter, we obtain that

$$\big\langle p, p+(1-\lambda)(a-b)\big\rangle + \eta(\alpha-\gamma+\eta) \le$$

$$\frac{|||(p,\eta)|||}{2r}\|p+(1-\lambda)(a-b)\|^2 = \tag{2.20}$$

$$\frac{|||(p,\eta)|||}{2r}\left(\|p\|^2+2(1-\lambda)\langle p,a-b\rangle+(1-\lambda)^2\|a-b\|^2\right).$$

Analogously, as

$$|||(b,\beta)-(u,\xi)|||\le|||(b,\beta)-(x,\gamma)|||+|||(x,\gamma)-(u,\xi)|||=$$
$$=\lambda|||(a,\alpha)-(b,\beta)|||+|||(x,\gamma)-(u,\xi)|||\le$$
$$\le\lambda|||(a,\alpha)-(b,\beta)|||+|||(x,\gamma)-(a,\alpha)|||=$$
$$\le\lambda|||(a,\alpha)-(b,\beta)|||+(1-\lambda)|||(a,\alpha)-(b,\beta)|||=$$
$$=|||(a,\alpha)-(b,\beta)|||<2r,$$

we can put $(x',\alpha')=(b,\beta)$ in (2.19) to obtain

$$\big\langle p,p+\lambda(b-a)\big\rangle+\eta(\beta-\gamma+\eta)\le$$

$$\frac{|||(p,\eta)|||}{2r}\left(\|p\|^2+2\lambda\langle p,b-a\rangle+\lambda^2\|a-b\|^2\right). \tag{2.21}$$

Multiplying (2.20) by λ, (2.21) by $(1-\lambda)$ and adding them we obtain

$$\langle p,p\rangle+\eta(\eta+(\lambda\alpha+(1-\lambda)\beta-\gamma))\le\frac{|||(p,\eta)|||}{2r}\left(\|p\|^2+\lambda(1-\lambda)\|a-b\|^2\right).$$

Since $\gamma=\lambda\alpha+(1-\lambda)\beta$, the latter yields

$$|||(p,\eta)|||^2=\|p\|^2+\eta^2\le\frac{|||(p,\eta)|||}{2r}\left(\|p\|^2+\lambda(1-\lambda)\|a-b\|^2\right).$$

Hence,

$$2r|||(p,\eta)|||\le\|p\|^2+\lambda(1-\lambda)\|a-b\|^2. \tag{2.22}$$

As $\|p\|\le|||(p,\eta)|||$ from the latter it holds that the quadratic inequality

$$t^2-2rt+\lambda(1-\lambda)\|a-b\|^2\ge 0. \tag{2.23}$$

is satisfied by $|||(p,\eta)|||$ as well as by $\|p\|$. Since $\|a-b\|\le|||(a,\alpha)-(b,\beta)|||<2r$, and $\lambda\in[0,1]$, for the discriminant of the left-hand side of this quadratic inequality we have

$$D:=4r^2-4\lambda(1-\lambda)\|a-b\|^2>0(\text{see }(1.9)).$$

So any t satisfying (2.23) should be such that $t\le t_1$ or $t\ge t_2$, where

$$t_1:=r-\sqrt{r^2-\lambda(1-\lambda)\|a-b\|^2},\quad t_2:=r+\sqrt{r^2-\lambda(1-\lambda)\|a-b\|^2}.$$

Since $t_2 \geq r > |||p, \eta||| \geq \|p\|$, we have that $\|p\| \leq t_1$, which reads

$$\|p\| \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} = \varphi(\lambda).$$

Using the latter in (2.22) we obtain the following

$$2r|||(p,\eta)||| \leq \left(r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}\right)^2 + \lambda(1-\lambda)\|a-b\|^2 =$$

$$= r^2 - 2r\sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} + r^2 - \lambda(1-\lambda)\|a-b\|^2 + \lambda(1-\lambda)\|a-b\|^2 =$$
$$2r^2 - 2r\sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2},$$

or finally

$$|||(p,\eta)||| \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} = \varphi(\lambda),$$

which is (2.17) and the proof is completed. □

Let us note here that if we had used only the prox-regularity of C, the estimate would be

$$d_C(x_\lambda, \gamma_\lambda) = |||(x_\lambda, \gamma_\lambda) - (u,\xi)||| \leq r - \sqrt{r^2 - \lambda(1-\lambda)|||(a,\alpha)-(b,\beta)|||^2},$$

which because of $|||(a,\alpha)-(b,\beta)||| \geq \|a-b\|$ is weaker than the estimate (2.17) we obtained.

With the following Theorem 2.3.2 we show that the converse is also true.

Theorem 2.3.2. *Let $C \subset \overline{H}$ be an epigraph set. Then the following are equivalent:*
(a) *C is epi r prox-regular;*
(b) *For any $(a,\alpha),(b,\beta) \in C$ such that $|||(a,\alpha)-(b,\beta)||| < 2r$, it holds that*

$$d_C\big(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta\big) \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2};$$

(c) *For any $(a,\alpha),(b,\beta) \in C$ such that $|||(a,\alpha)-(b,\beta)||| < 2r$, it holds that*

$$d_C\big(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta\big) \leq \frac{1}{2r}\min\{\lambda, 1-\lambda\}\|a-b\|^2. \tag{2.24}$$

Proof. The statements will be proved in the following order

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a).$$

$(a) \Rightarrow (b)$ The implication (a)⇒(b) is established in Theorem 2.3.1.

$(b) \Rightarrow (c)$ Let (b) holds. To prove (b)⇒(c) it is enough to show that if $\|a-b\| < 2r$ and $\lambda \in [0,1]$, then

$$r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} \leq \frac{1}{2r}\min\{\lambda, 1-\lambda\}\|a-b\|^2.$$

To this end we consider the following two cases.

CASE 1. $\min\{\lambda, 1-\lambda\} = \lambda$. In this case we have to show that

$$r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} \leq \frac{\lambda}{2r}\|a-b\|^2,$$

or

$$r - \frac{\lambda}{2r}\|a-b\|^2 \leq \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Since $\lambda \leq \frac{1}{2}$ the left hand side of the inequality is greater than zero, so it is equivalent to show that

$$\left(r - \frac{\lambda}{2r}\|a-b\|^2\right)^2 \leq r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

or

$$r^2 - \lambda\|a-b\|^2 + \frac{\lambda^2}{4r^2}\|a-b\|^4 \leq r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

or

$$\big(\lambda(1-\lambda) - \lambda\big)\|a-b\|^2 + \frac{\lambda^2}{4r^2}\|a-b\|^4 \leq 0,$$

or finally

$$\lambda^2\|a-b\|^2\left(\frac{\|a-b\|^2}{4r^2} - 1\right) \leq 0.$$

The last inequality holds since $\|a-b\| < 2r$.

CASE 2. $\min\{\lambda, 1-\lambda\} = 1-\lambda$. In this case and we have to show that

$$r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} \leq \frac{1-\lambda}{2r}\|a-b\|^2,$$

or equivalently

$$r - \frac{1-\lambda}{2r}\|a-b\|^2 \leq \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Since $1-\lambda \leq \frac{1}{2}$ the left hand side of the inequality is positive, so it is equivalent to show that

$$\left(r - \frac{1-\lambda}{2r}\|a-b\|^2\right)^2 \leq r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

or

$$r^2 - (1-\lambda)\|a-b\|^2 + \frac{(1-\lambda)^2}{4r^2}\|a-b\|^4 \leq r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

or

$$\big(\lambda(1-\lambda) - (1-\lambda)\big)\|a-b\|^2 + \frac{(1-\lambda)^2}{4r^2}\|a-b\|^4 \leq 0,$$

or finally

$$(1-\lambda)^2\|a-b\|^2\left(\frac{\|a-b\|^2}{4r^2}-1\right)\leq 0,$$

which holds as $\|a-b\|<2r$. The proof of (b)⇒(c) is completed.

$(c)\Rightarrow(a)$ Let (c) holds and $y=(a,\alpha)$, $x=(b,\beta)\in C$ be such that $|||x-y|||<2r$, and $v=(q,\eta)\in N_C(x)\cap\mathbb{B}_{\overline{H}}$.

We consider $(H\times\mathbb{R})\times\mathbb{R}$ with the norm $||||\cdot||||$ defined as

$$||||(z,\alpha)|||| := \sqrt{|||z|||^2+|\alpha|^2}.$$

Hence, $(\overline{H}\times\mathbb{R}, ||||\cdot||||)$ is a Hilbert space.

Since $v\in N_C(x)\cap\mathbb{B}_{\overline{H}}$ we have that $(v,-1)\in N_{\operatorname{epi} d_C}(x,d_C(x))\subset\overline{H}\times\mathbb{R}$ (see Proposition A.1.9 in the Appendix) and then by (1.3) there exist some $\sigma>0$ such that

$$\langle(v,-1)\,,(x'-x,\alpha'-d_C(x))\rangle\leq\sigma||||(x'-x,\alpha'-d_C(x))||||^2,\quad\forall(x',\alpha')\in\operatorname{epi} d_C.$$

Since $x\in C$, we have that $d_C(x)=0$ and

$$\langle(v,-1)\,,(x'-x,\alpha')\rangle\leq\sigma||||(x'-x,\alpha')||||^2,\quad\forall(x',\alpha')\in\operatorname{epi} d_C.$$

For $\lambda\in[0,1]$ consider the point

$$z_\lambda := \lambda x+(1-\lambda)y.$$

For the point $(x',\alpha')=(z_\lambda,d_C(z_\lambda))$ the last inequality gives

$$\langle v,z_\lambda-x\rangle-d_C(z_\lambda)\leq\sigma\left(|||z_\lambda-x|||^2+d_C^2(z_\lambda)\right).$$

Hence,

$$\begin{aligned}\langle v,z_\lambda-x\rangle &\leq d_C(z_\lambda)+\sigma|||z_\lambda-x|||^2+\sigma d_C^2(z_\lambda)\\ &\leq d_C(z_\lambda)+\sigma|||z_\lambda-x|||^2+\sigma|||z_\lambda-x|||^2\\ &= d_C(z_\lambda)+2\sigma|||z_\lambda-x|||^2.\end{aligned}\tag{2.25}$$

So, for $\lambda<1/2$, we have

$$\begin{aligned}\langle v,\lambda(a-b,\alpha-\beta)\rangle &= \langle v,z_\lambda-x\rangle\quad\text{from (2.25)}\\ &\leq d_C(z_\lambda)+2\sigma|||z_\lambda-x|||^2\\ &= d_C(z_\lambda)+2\sigma\lambda^2|||(b-a,\beta-\alpha)|||^2\quad\text{from (2.24)}\\ &\leq \frac{\lambda}{2r}\|a-b\|^2+2\sigma\lambda^2|||(b-a,\beta-\alpha)|||^2.\end{aligned}$$

Dividing the last inequality by $\lambda > 0$ we get

$$\langle v, (a-b, \alpha-\beta)\rangle \leq \frac{1}{2r}\|a-b\|^2 + 2\sigma\lambda |||(b-a, \beta-\alpha)|||^2.$$

Now letting λ to zero and using that $v = (q, \eta)$ we obtain

$$\langle (q,\eta), (a-b, \alpha-\beta)\rangle \leq \frac{1}{2r}\|a-b\|^2,$$

which entails (a). The proof is complete. □

2.4 Epigraphical characterization

Theorem 2.4.1. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. If f is epi r lower regular, then*
(i) *for any (a, α), $(b, \beta) \in \operatorname{epi} f$ such that*

$$|||(a,\alpha) - (b,\beta)||| < 2r$$

and any $\lambda \in [0,1]$ there is $(u_\lambda, \xi_\lambda) \in \operatorname{epi} f$ such that

$$\|u_\lambda - (\lambda a + (1-\lambda)b)\|^2 + |\xi_\lambda - (\lambda\alpha + (1-\lambda)\beta)|^2 \leq \varphi^2(\lambda), \tag{2.26}$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Conversely, if (i) *holds, then f is epi ρ lower regular for $\rho = \frac{r}{\sqrt{2}}$.*

Proof. Let f be an r lower regular function. According to Theorem 2.2.1, the set $C \equiv \operatorname{epi} f$ is epi r prox-regular in $\overline{H}$. Applying Theorem 2.3.1 to the set C and the points (a, α) and (b, β) in C we have that for any $\lambda \in [0,1]$ for $(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta)$, there exists $(u_\lambda, \xi_\lambda) \in C$ such that $|||(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta) - (u_\lambda, \xi_\lambda)||| \leq \varphi(\lambda)$, which is (2.26) and (i) holds.

It is clear that either $f(u_\lambda) \leq \lambda f(a) + (1-\lambda)f(b)$, or

$$\lambda f(a) + (1-\lambda)f(b) < f(u_\lambda) \leq \xi_\lambda \leq \lambda\alpha + (1-\lambda)\beta + \varphi(\lambda).$$

Let now (i) holds for f. This is equivalent to the feature that the epigraph set $C \equiv \operatorname{epi} f$ satisfies the condition of Theorem 2.3.2(b) and, therefore, it is an epi r prox-regular set. Then Theorem 2.2.2 ensures that f is an epi ρ lower regular function for $\rho = \frac{r}{\sqrt{2}}$. □

Taking $\alpha = f(a)$, $\beta = f(b)$ in Theorem 2.4.1 we obtain

Corollary 2.4.2. *If* $f : H \to \mathbb{R} \cup \{+\infty\}$ *is* ρ *primal lower regular function, then for any* $a, b \in \operatorname{dom} f$ *such that* $|||(a, f(a)) - (b, f(b))||| < 2r$ *and any* $\lambda \in [0, 1]$ *there is* $u \in \operatorname{dom} f \cap B[\lambda a + (1-\lambda)b), \varphi(\lambda)]$ *such that either*

$$f(u) \leq \lambda f(a) + (1-\lambda) f(b),$$

or

$$\lambda f(a) + (1-\lambda) f(b) < f(u) \leq \lambda f(a) + (1-\lambda) f(b) + \varphi(\lambda),$$

where $\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}$*. In particular,*

$$\inf_{B[\lambda a + (1-\lambda)b), \varphi(\lambda)]} f \leq \lambda f(a) + (1-\lambda) f(b) + \varphi(\lambda).$$

Theorem2.4.1 says that *Convex-like* functions f such that for some $r > 0$ it holds that for any $(a, \alpha), (b, \beta) \in \operatorname{epi} f$ with $||||(a, \alpha) - (b, \beta)|||| < 2r$ and $\lambda \in (0, 1)$ there exist $(u_\lambda, \xi_\lambda) \in \operatorname{epi} f$ such that

$$\|u_\lambda - (\lambda a + (1-\lambda)b)\|^2 + |\xi_\lambda - (\lambda \alpha + (1-\lambda)\beta)|^2 \leq \varphi^2(\lambda)$$

with

$$\varphi(\lambda) = r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}$$

are exactly the functions which proximal subdifferential for some $r > 0$ has the property

$$\alpha' \geq f(x) + \langle p, x' - x \rangle - \frac{t}{2}\|x' - x\|^2$$

for all $(x', \alpha') \in \mathbb{B}((x, f(x)), 2\rho) \cap \operatorname{epi} f$ where $t \geq r^{-1}$ and $p \in \partial^p f(x)$.

Chapter 3

Epsilon Subdifferential Method and Integrability

The Epsilon Subdifferential Method is well known and widely used for minimizing convex functions, see e.g. [12, 13]. In this chapter we develop a novel Epsilon Subdifferential Method (ESM).

We will outline it here:

ESM applies to a given proper, convex and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined on a Banach space X, such that

$$0 = f(0) = \min_{x \in X} f(x)$$

with fixed in advance parameters $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$.

Starting at an arbitrary $x_0 \in \operatorname{dom} f$, for $i = 0, 1, \dots$

- if $0 \in \partial_\varepsilon f(x_i)$, then STOP;
- if $0 \notin \partial_\varepsilon f(x_i)$, for

  $$\varphi_{x_i}(K) := \inf_{x \in X} F_{x_i}(K, x),$$

  where

  $$F_{x_i}(K, x) := f(x) - f(x_i) + \varepsilon + K\|x - x_i\|,$$

  find $K_i > 0$ such that $\varphi_i(K_i) = 0$.

  Take any x_{i+1} satisfying

  $$0 \leq f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \leq \delta.$$

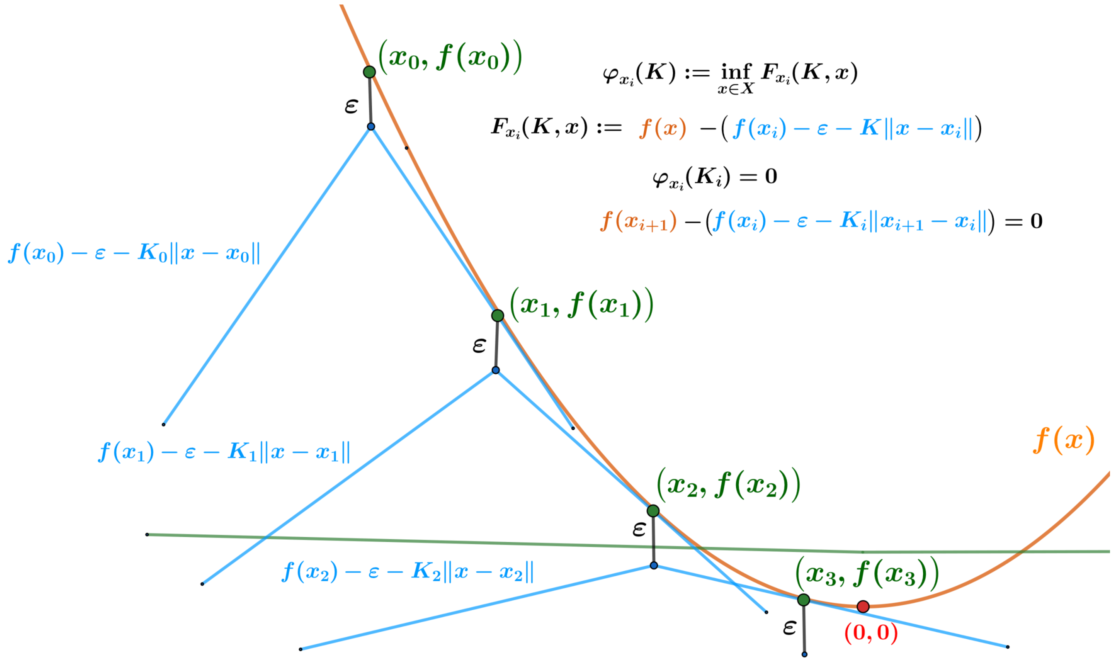

Figure 3.1: Visualization of the Novel ESM in the one dimensional case

In the finite dimensional case $\delta = 0$ works and our novel ESM is much more simple, i.e. x_{i+1} is the unique solution to the equation

$$f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| = 0.$$

For the one dimensional case see the illustration in Figure 3.1.

Returning to the Banach space case if for some $c > 0$ the function f satisfies

$$f(x) \geq c\|x\| \text{ for all } x \in X,$$

the parameter δ is appropriately chosen, and the starting point $x_0 \in \operatorname{dom} \partial f$, then the number of iterations n of our novel ESM can be estimated by

$$n\sqrt{\varepsilon} \leq \text{const},$$

(see Lemma 3.2.3). The proof of this estimate relies on Lemma 3.1.5.

Note that the immediate estimate provided by the classical method is

$$n\varepsilon \leq \text{const}.$$

So, in our case $n\varepsilon$ tends to 0 as ε tends to 0, which allows us to present a new prove of the Moreau-Rockafellar Theorem, see e.g. [45, 46]:

Theorem 3.1.1. *Let X be a Banach space. Let g and h be proper, lower semicontinuous and convex functions from X to $\mathbb{R} \cup \{+\infty\}$. If*

$$\partial g \subset \partial h, \tag{3.1}$$

then

$$h = g + \text{const}.$$

This result has numerous important implications, see e.g. Section 3 of Phelps' book [40].

Let us make a short historical overview. The integrability of the subdifferential of proper lower semicontinuous convex function on Hilbert space is stated and proved first by Moreau in [37] by using Moreau-Yosida regularisation. The proof also works in reflexive Banach space as mentioned at p. 87 of [38]. The first complete proof in Banach space – that of Rockafellar in [46] – uses strong duality arguments. Another approach is to approximate the directional derivative and to reduce to the one-dimensional case. The latter was taken by Rockafellar in his original proof in [45]. Though there are some gaps in this proof, Taylor [48] fills them and provides a different proof, cf. [14]. The idea of directional derivative approximation/one dimensional reduction is most clearly outlined in the proof of Thibault [49]. A different proof using the mean-value theorem of Zagrodny is due to Thibault and Zagrodny [50], see also [54]. In [55] the result is proved by using regularization (and approximation) techniques which was the initial idea of Moreau.

In [29] Ivanov and Zlateva give a proof similar to the proof of the classical calculus theorem that a monotone function is Riemann integrable which uses neither duality nor explicit one-dimensional arguments. The main step in their proof is to show directly that a proper lower semicontinuous convex function on Banach space differs by a constant from the *Rockafellar function* (see [7]) of its subdifferential, see [29, Theorem 1.2]. The proof relies on a technical [29, Lemma 3.3] proved by Ekeland variational principle.

Here we use the novel ESM to prove in a different way the following

Theorem 3.1.2 (Rockafellar [45, 46], see also [29] Theorem 1.2)**.** *Let*

$$g : X \to \mathbb{R} \cup \{+\infty\}$$

be a proper, lower semicontinuous and convex function. Let $\bar{x} \in \operatorname{dom} \partial g$ and $\bar{p} \in \partial g(\bar{x})$. Then for all $x \in X$

$$g(x) = g(\bar{x}) + R_{\partial g, (\bar{x}, \bar{p})}(x),$$

where

$$R_{\partial g, (\bar{x}, \bar{p})}(x) := \sup \Big\{ \sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle : \tag{3.2}$$

$$x_0 = x,\ x_n = \bar{x},\ q_n = \bar{p},\ q_i \in \partial g(x_i), n \in \mathbb{N} \Big\}.$$

A distinctive feature of the new proof here is that it reveals the relationship between a natural optimization method and the Moreau-Rockafellar Theorem. By use of ESM the sequences realizing supremum in (3.2) are kind of constructed.

Thereafter, the proof of Theorem 3.1.1 continues exactly as in [29]. That is why, we only sketch it here: it readily follows that (3.1) implies

$$g(x) - g(\overline{x}) \leq h(x) - h(\overline{x})$$

for any $\overline{x} \in \operatorname{dom} \partial g$ and all $x \in X$. In particular, $g - h \equiv \text{const}$ on $\operatorname{dom} \partial g$. To conclude, we use lower semicontinuity of h and graphical density of points of subdifferentiability to g, i.e. that for any $\bar{x} \in \operatorname{dom} g$ and any $\varepsilon > 0$ there exists $x \in \operatorname{dom} \partial g$ such that $\|x - \bar{x}\| + |g(x) - g(\bar{x})| < \varepsilon$, see [18] and [14].

Let us also note that tools used in the proof had been known by 1970.

After a short Section 3.1 on notations and preliminaries, in Section 3.2 we dwell on some of the basic properties of our novel Epsilon Subdifferential Method. In the last Section 3.3 we give the proof of Theorem 3.1.2.

The results in this chapter are published in [32].

3.1 Introductory notations

The notations used throughout the chapter are standard. $(X, \|\cdot\|)$ denotes a real Banach space, that is, a complete normed space over $\mathbb{R}$. The dual space X^* of X is the Banach space of all continuous linear functionals p from X to $\mathbb{R}$. The natural norm of X^* is again denoted by $\|\cdot\|$.

The value of $p \in X^*$ at $x \in X$ is denoted by $\langle p, x \rangle$.

Let us recall that for $\varepsilon \geq 0$, the ε-subdifferential of a proper, lower semicontinuous and convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is the set

$$\partial_\varepsilon f(x) := \{p \in X^* \ : \ -\varepsilon + \langle p, y - x \rangle \leq f(y) - f(x), \quad \forall y \in X\},$$

and $\partial_\varepsilon f = \varnothing$ on $X \setminus \operatorname{dom} f$. Of course, for $\varepsilon = 0$, $\partial_0 f(x)$ coincides with the subdifferential of f at x in the sense of Convex Analysis $\partial f(x)$.

The *domain* $\operatorname{dom} \partial_\varepsilon f$ consists of all points $x \in X$ such that $\partial_\varepsilon f(x)$ is non-empty. But while $\partial f(x)$ could be empty, for $\varepsilon > 0$, the sets $\partial_\varepsilon f(x)$ are non-empty for any $x \in \operatorname{dom} f$.

For any real numbers $\varepsilon_1$and ε_2 such that $0 < \varepsilon_1 \leq \varepsilon_2$ one has

$$\partial_{\varepsilon_1} f(x) \subset \partial_{\varepsilon_2} f(x) \text{ and } \partial f(x) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(x).$$

Moreover, if $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are two proper lower semicontinuous convex functions with $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and one of them is continuous at x, then the following sum rule holds, see e.g. [54, Theorem 2.8.7],

$$\partial_\varepsilon(f+g)(x) = \bigcup\{\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) : \varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0\ \varepsilon = \varepsilon_1 + \varepsilon_2\}.$$

We will use it further in its weaker form

$$\partial_\varepsilon(f+g)(x) \subset \partial_\varepsilon f(x) + \partial_\varepsilon g(x).$$

The result of Brøndsted and Rockafellar saying that the graph of $\partial_\varepsilon f$ is close to the graph of ∂f is well known:

Theorem 3.1.3 (Brøndsted-Rockafellar [18])**.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, be a proper, convex and lower semicontinuous function, let $\varepsilon > 0$ and $p \in \partial_\varepsilon f(x)$. Then there exists $q \in \partial f(z)$ such that*

$$\|z - x\| \leq \sqrt{\varepsilon}, \text{ and } \|q - p\| \leq \sqrt{\varepsilon}.$$

Another result of Brøndsted and Rockafellar [18] also will be used:

Proposition 3.1.4. *Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Then for all $x \in X$*

$$f(x) = \sup\{f(\bar{x}) + \bar{p}(x - \bar{x});\ (\bar{x}, \bar{p}) \in \operatorname{gph} \partial f\}. \tag{3.3}$$

To estimate the number of iteration of the novel ESM we prove the following result of its own interest.

Lemma 3.1.5. *Let $n \in \mathbb{N}$ and $A > 0$, $B > 0$, $\varepsilon > 0$ be real numbers. If there exist reals $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ which satisfy the following conditions*

$$a_i > 0 \text{ and } b_i > 0 \text{ for all } i \in \{1, \ldots, n\}, \tag{3.4}$$

$$a_i\, b_i \geq \varepsilon \text{ for all } i \in \{1, \ldots, n\}, \tag{3.5}$$

$$\sum_{i=1}^{n} a_i \leq A, \tag{3.6}$$

$$\sum_{i=1}^{n} b_i \leq B, \tag{3.7}$$

then the inequality

$$n \leq \sqrt{\frac{AB}{\varepsilon}}$$

holds.

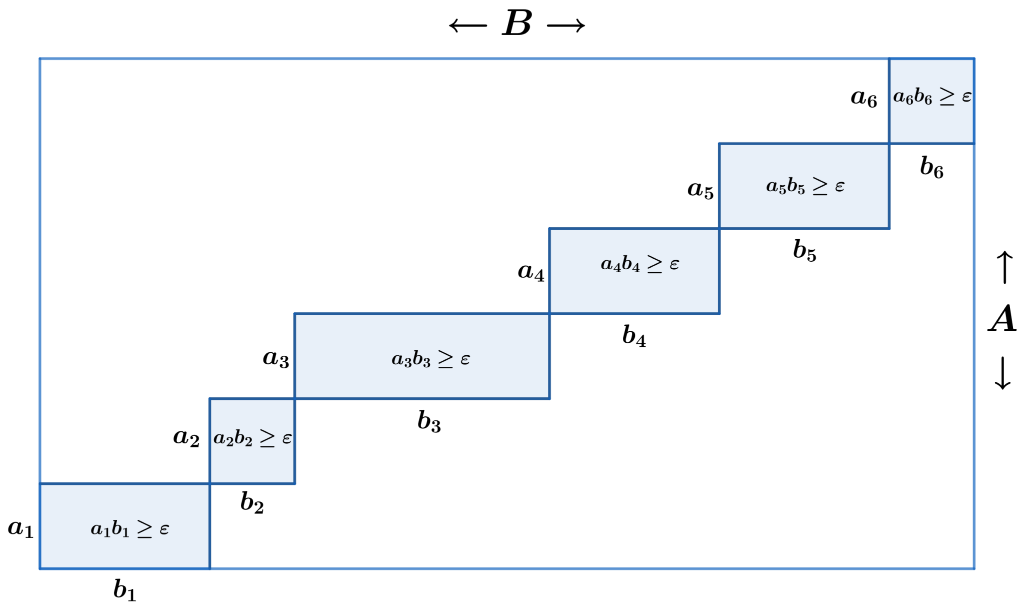

Figure 3.2: Geometric interpretation of Lemma 3.1.5

Proof. From (3.4) and (3.5) follow the inequalities

$$b_i \geq \frac{\varepsilon}{a_i} \text{ for all } i \in \{1, \ldots, n\}. \tag{3.8}$$

Summing the inequalities (3.8) for all $i \in \{1, \ldots, n\}$ we get by (3.7) that

$$\sum_{i=1}^{n} \frac{\varepsilon}{a_i} \leq \sum_{i=1}^{n} b_i \leq B.$$

This means that $\sum_{i=1}^{n} \frac{1}{a_i} \leq \frac{B}{\varepsilon}$ and, equivalently, $\frac{\varepsilon}{B} \leq \left(\sum_{i=1}^{n} \frac{1}{a_i} \right)^{-1}$.
Multiplying by n in the later we get

$$\frac{n\varepsilon}{B} \leq n \left(\sum_{i=1}^{n} \frac{1}{a_i} \right)^{-1}. \tag{3.9}$$

From (3.6) it follows that

$$\frac{\sum_{i=1}^{n} a_i}{n} \leq \frac{A}{n}. \tag{3.10}$$

By (3.9), the Cauchy inequality between the harmonic mean and arithmetic mean and (3.10) we get the following chain of inequalities

$$\frac{n\varepsilon}{B} \leq \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \frac{\sum_{i=1}^{n} a_i}{n} \leq \frac{A}{n}.$$

which yields that $n^2 \leq \frac{AB}{\varepsilon}$. Therefore, $n \leq \sqrt{\frac{AB}{\varepsilon}}$. □

Note that the expression $\sqrt{\frac{AB}{\varepsilon}}$ can be interpreted as an upper bound of the number n of rectangles with face at least ε and sides respectively (a_1, b_1), (a_2, b_2) $\ldots (a_n, b_n)$ which can be placed next to each other without intersection on the diagonal of fixed rectangles with sides (A, B), see Figure 3.2.

3.2 Novel Epsilon Subdifferential Method

We have outlined the method in the beginning of the chapter. In this section we will consider some of its properties. Throughout this section we work with a proper, lower semicontinuous and convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, such that

$$\min_{x \in X} f(x) = f(0) = 0,$$

and fixed $\varepsilon > 0$, and $\varepsilon > \delta > 0$.

The next result describes what happens at one of our Novel ESM iterations.

Lemma 3.2.1. *Let $x_0 \in \operatorname{dom} f$. The function $\varphi_{x_0} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_{x_0}(K) := \inf_{x \in X} F_{x_0}(K, x),$$

where

$$F_{x_0}(K, x) := f(x) - f(x_0) + \varepsilon + K\|x - x_0\|,$$

is strictly monotone increasing and locally Lipschitz on $(0, \infty)$.

Assume in addition that $0 \notin \partial_\varepsilon f(x_0)$. Then

(i) there exists $K_0 > 0$ such that $\varphi_{x_0}(K_0) = 0$;

(ii) for any $x_1 \in X$ *such that*

$$0 \le f(x_1) - f(x_0) + \varepsilon + K_0\|x_1 - x_0\| \le \delta, \tag{3.11}$$

there is $p_1 \in \partial_\delta f(x_1)$ *such that*

$$K_0 \ge \|p_1\| - \delta, \tag{3.12}$$

and,

$$\langle p_1, x_1 - x_0\rangle \le f(x_1) - f(x_0) + \varepsilon + \delta. \tag{3.13}$$

Moreover,

$$K_0 \le \min\{\|p\| : p \in \partial_\varepsilon f(x_0)\}, \tag{3.14}$$

and if $p_0 \in \partial_\delta f(x_0)$*, then*

$$\varepsilon \le (\|p_0\| - \|p_1\|)\|x_1 - x_0\| + \delta\left(2 + \frac{f(x_0)}{K_0}\right). \tag{3.15}$$

Proof. It is straightforward to see that the function φ_{x_0} is strictly monotonic on $(0,\infty)$. For the proof of the local Lipschitz continuity of φ_{x_0} on $(0,\infty)$ we refer to Proposition A.1.10 in the Appendix. To establish (i), it is enough to show that there are $\widetilde{K}_0, \widehat{K}_0 > 0$ such that $\varphi_{x_0}(\widetilde{K}_0) < 0$ and $\varphi_{x_0}(\widehat{K}_0) > 0$ and then apply the Theorem of Bolzano. For the proof of the latter we refer to Proposition A.1.11 in the Appendix.

Now, let $x_1 \in X$ be a point of δ-infimum of the function $F_{x_0}(K_0,\cdot)$, i.e. satisfying (3.11). Equivalently,

$$0 \in \partial_\delta\left(f(\cdot) - f(x_0) + \varepsilon + K_0\|\cdot - x_0\|\right)(x_1).$$

By the weaker form of the sum rule for the δ-subdifferential, we have that there exist $p_1 \in \partial_\delta f(x_1)$, and $\xi_1 \in \partial_\delta K_0\|\cdot - x_0\|(x_1)$ such that $0 = p_1 + \xi_1$. Since $\xi_1 \in \partial_\delta K_0\|\cdot - x_0\|(x_1)$,

$$\langle \xi_1, x - x_1\rangle \le K_0\|x - x_0\| - K_0\|x_1 - x_0\| + \delta \le K_0\|x - x_1\| + \delta,\ \forall x \in X. \tag{3.16}$$

Hence,

$$|\langle \xi_1, x - x_1\rangle| \le K_0\|x - x_1\| + \delta,\ \forall x \in X,$$

and (3.12) holds. From (3.16) easily follows that

$$K_0\|x_1 - x_0\| \le \langle \xi_1, x_1 - x_0\rangle + \delta = \langle p_1, x_0 - x_1\rangle + \delta$$

which combined with the left inequality in (3.11) yields (3.13). Take arbitrary $p \in \partial_\varepsilon f(x_0)$. By definition of the ε-subdifferential,

$$\langle p, x - x_0\rangle \le f(x) - f(x_0) + \varepsilon,\ \forall x \in X.$$

Hence,

$$\langle p, x - x_0 \rangle + K_0\|x - x_0\| \leq f(x) - f(x_0) + \varepsilon + K_0\|x - x_0\|, \ \forall x \in X,$$

$$\langle p, \frac{x - x_0}{\|x - x_0\|} \rangle + K_0 \leq \frac{f(x) - f(x_0) + \varepsilon + K_0\|x - x_0\|}{\|x - x_0\|}, \ \forall x \in X, \ x \neq x_0,$$

$$K_0 + \inf_{x \in X, x \neq x_0} \langle p, \frac{x - x_0}{\|x - x_0\|} \rangle \leq \inf_{x \in X, x \neq x_0} \left(\frac{f(x) - f(x_0) + \varepsilon + K_0\|x - x_0\|}{\|x - x_0\|} \right) = 0.$$

Finally, $K_0 \leq \|p\|$, and (3.14) holds. Take any $p_0 \in \partial_\delta f(x_0)$. Using (3.12), and $\|x_1 - x_0\| \leq \dfrac{f(x_0)}{K_0}$ (which is an easy consequence of (3.11) and $\delta < \varepsilon$), we get that

$$\begin{aligned}
\varepsilon &\leq f(x_0) - f(x_1) - K_0\|x_1 - x_0\| + \delta \\
&\leq \langle p_0, x_0 - x_1 \rangle - \|p_1\|\|x_1 - x_0\| + \delta\|x_1 - x_0\| + 2\delta \\
&\leq \|p_0\|\|x_1 - x_0\| - \|p_1\|\|x_1 - x_0\| + \delta \left(\frac{f(x_0)}{K_0} + 2 \right) \\
&= (\|p_0\| - \|p_1\|)\|x_1 - x_0\| + \delta \left(\frac{f(x_0)}{K_0} + 2 \right),
\end{aligned}$$

which is (3.15). The proof is then completed. □

In the context of the ESM, Lemma 3.2.1 ensures the existence of $K_i > 0$. As x_{i+1} can be taken any point of δ-minimum, i.e. such that

$$0 \leq f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \leq \delta. \tag{3.17}$$

From the lemma we also have the existence of $p_{i+1} \in \partial_\delta f(x_{i+1})$ such that

$$K_i \geq \|p_{i+1}\| - \delta, \ i \geq 0, \tag{3.18}$$

$$\langle p_{i+1}, x_{i+1} - x_i \rangle \leq f(x_{i+1}) - f(x_i) + \varepsilon + \delta, \ i \geq 0, \tag{3.19}$$

as well as,

$$\varepsilon \leq (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\| + \delta \left(2 + \frac{f(x_i)}{K_i} \right), \ i \geq 1. \tag{3.20}$$

The next Lemma shows that our Novel ESM is finite.

Lemma 3.2.2. *The novel ESM ends after a finite number of iterations n such that $n \leq \dfrac{f(x_0)}{\varepsilon - \delta} + 1$ at point x_{n-1} of ε-minimum of f.*

Proof. Let us assume the contrary, i.e. that the number of iterations satisfy $n > \dfrac{f(x_0)}{\varepsilon - \delta} + 1$ and fix such $n \in \mathbb{N}$. This means that ESM generates at least x_i, $i = 0, \dots, n-1$ such that

$$0 \notin \partial_\varepsilon f(x_i), \quad i = 0, \dots, n-2.$$

Then from (3.17) we will have that

$$f(x_i) - f(x_{i+1}) \geq \varepsilon + K_i \|x_{i+1} - x_i\| - \delta \geq \varepsilon - \delta > 0, \qquad i = 0, \dots, n-2.$$

Summing the inequalities we obtain that

$$f(x_0) - f(x_{n-1}) = \sum_{i=0}^{n-2} \left(f(x_i) - f(x_{i+1})\right) \geq (n-1)(\varepsilon - \delta) > f(x_0),$$

hence $0 > f(x_{n-1})$ which contradicts to $f(x_{n-1}) \geq f(0) = 0$. □

It is possible to obtain a better estimate of the number of iteration for a strictly convex function with more precise choice of the parameter δ.

Lemma 3.2.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function satisfying $f(x) \geq 2c\|x\|$ for all $x \in X$ and some $c > 0$.*

Applied for f with $\varepsilon > 0$ and $\delta > 0$ such that

$$\delta \leq \frac{c}{2}, \quad \delta \leq 1, \quad \delta \left(1 + \frac{f(x_0)}{c}\right) \leq \frac{\varepsilon}{4}, \tag{3.21}$$

ESM ends after n iterations, and

$$\sum_{i=0}^{n-2} \|x_{i+1} - x_i\| \leq \frac{2f(x_0)}{c}. \tag{3.22}$$

Moreover, for the number of iterations n we have the estimation

$$n \leq 2\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c\varepsilon}} + 2, \tag{3.23}$$

where $p_0 \in \partial_\varepsilon f(x_0)$ is arbitrary.

Proof. Since $f(x) \geq 2c\|x\|$, it is easy to see that if $p \in \partial_\delta f(x)$, then

$$0 = f(0) \geq f(x) - \langle p, x \rangle - \delta \geq 2c\|x\| - \langle p, x \rangle - \delta$$

yields

$$\|p\|\|x\| \geq \langle p, x \rangle \geq 2c\|x\| - \delta. \tag{3.24}$$

We have three cases: (a) $\|p\| \geq 2c$; (b) $\|p\| < 2c$, and $\|x\| > \delta/c$, and (c) $\|p\| < 2c$, and $\|x\| \leq \delta/c$.

In case (b), by (3.24) we have

$$\|p\| \geq 2c - \frac{\delta}{\|x\|} > 2c - \frac{\delta c}{\delta} = c.$$

In case (c),

$$f(x) \leq \langle p, x\rangle + \delta \leq \|p\|\|x\| + \delta \leq 3\delta < \varepsilon,$$

and x should be a point of ε-minimum for f.

As x_i, $i = 0, \dots, n-2$, are not ε-minimum points for f, the latter implies, see (3.18), that

$$K_i \geq \|p_{i+1}\| - \delta \geq c - \delta \geq c - \frac{c}{2} = \frac{c}{2}.$$

To establish (3.22) we sum up inequalities (3.17) from 0 to $n-2$ to get that

$$f(x_{n-1}) - f(x_0) + (n-1)\varepsilon + \sum_{i=0}^{n-2} K_i\|x_{i+1} - x_i\| \leq (n-1)\delta.$$

Hence,

$$\sum_{i=0}^{n-2} K_i\|x_{i+1} - x_i\| + (n-1)(\varepsilon - \delta) \leq f(x_0) - f(x_{n-1}).$$

Since $K_i \geq \dfrac{c}{2}$ for all i in the above sum, and $\delta < \varepsilon$,

$$\frac{c}{2}\sum_{i=0}^{n-2} \|x_{i+1} - x_i\| \leq f(x_0),$$

and (3.22) holds.

Since $f(x_{i+1}) \leq f(x_i)$ for all i, see (3.17), we have that $f(x_i) \leq f(x_0)$ for all i. Using this and $K_i \geq \dfrac{c}{2}$ in (3.20) we obtain that

$$\varepsilon \leq (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\| + 2\delta\left(1 + \frac{f(x_0)}{c}\right), \ i \geq 1,$$

hence, having in mind the choice of δ,

$$\frac{\varepsilon}{2} \leq (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\|, \ i \geq 1. \tag{3.25}$$

To apply Lemma 3.1.5, set

$$a_i := \|p_i\| - \|p_{i+1}\|, \quad b_i := \|x_{i+1} - x_i\|, \quad i = 1, \dots, n-2.$$

From (3.25) we have that $a_i b_i \geq \varepsilon/2$, hence, $a_i > 0$, $b_i > 0$, $i = 1, \dots, n-2$. From (3.22)

$$\sum_{i=1}^{n-2} b_i \leq \frac{2f(x_0)}{c}.$$

On the other hand,

$$\sum_{i=1}^{n-2} a_i = \|p_1\| - \|p_{n-1}\| \leq \|p_1\| \leq K_0 + \delta \leq \|p_0\| + \delta \leq \|p_0\| + 1,$$

where $p_0 \in \partial_\varepsilon f(x_0)$ is arbitrary (see (3.14)).

Setting $A := \|p_0\| + 1$ and $B := \dfrac{2f(x_0)}{c}$ we have that the conditions of Lemma 3.1.5 hold. Hence,

$$n - 2 \leq \sqrt{\frac{2AB}{\varepsilon}} = 2\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c\varepsilon}}$$

and (3.23) holds. The proof is completed. □

Let us note that p_0 in (3.23) as an arbitrary element in $\partial_\varepsilon f(x_0)$ depends on ε. But when $x_0 \in \operatorname{dom} \partial f$, then p_0 could be taken in $\partial f(x_0)$ and in this case, the estimation (3.23) is of the type $n\sqrt{\varepsilon} \leq \text{const}$.

3.3 New proof of Moreau-Rockafellar Theorem

In this section we present our novel proof of Theorem 3.1.2.

First we will establish

$$g(x) = g(\bar{x}) + R_{\partial g,(\bar{x},\bar{p})}(x) \text{ for all } x \in \operatorname{dom} \partial g.$$

It is easy to prove that

$$g(x) - g(\bar{x}) \geq R_{\partial g,(\bar{x},\bar{p})}(x). \tag{3.26}$$

Indeed, for any sequence $\{(x_i, q_i)\}_{i=1}^n \subset \operatorname{gph} \partial g$ with $x_0 = x$, $x_n = \bar{x}$, and $q_n = \bar{p}$, by the definition of subdifferential we have that

$$\langle q_{i+1}, x_i - x_{i+1} \rangle \leq g(x_i) - g(x_{i+1}), \quad i = 0, \dots, (n-1).$$

After summing these we immediately get

$$\sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle \leq g(x) - g(\bar{x})$$

and (3.26) follows.

To obtain that

$$g(x) - g(\bar{x}) \le R_{\partial g,(\bar{x},\bar{p})}(x)$$

it is enough for any fixed $\varepsilon' > 0$ to find a sequence $\{(x_i, q_i)\}_{i=1}^n \subset \operatorname{gph} \partial g$ such that $x_0 = x$, $x_n = \bar{x}$, $q_n = \bar{p}$, and

$$g(x) - g(\bar{x}) - \sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle < \varepsilon'. \tag{3.27}$$

To this end we consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined as

$$f(x) := g(x + \bar{x}) - \langle \bar{p}, x \rangle - g(\bar{x}) + 2c\|x\|, \tag{3.28}$$

where $c > 0$. It is easy to see that f is proper lower semicontinuous and convex, $f(0) = 0$, $0 \in \partial f(0)$, $f(x) \ge 2c\|x\|$ for all $x \in X$ and $\operatorname{dom} \partial f \equiv \operatorname{dom} \partial g - \bar{x}$.

Take an arbitrary $p_0 \in \partial f(x_0)$ and set

$$M := 4 \left(\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c}} + 1 \right).$$

Take $\varepsilon \in (0, 1)$ such that $M\sqrt{\varepsilon} < \varepsilon'$ and then apply ESM for f with this ε and $\delta > 0$ such that $\eta(\delta) < \varepsilon/3$, where

$$\eta(\delta) := 2\sqrt{\delta} \left(1 + 2c + \|p_0\| + \|\overline{p}\| + \frac{f(x_0)}{c} \right).$$

It is easy to check that if δ is such that $\eta(\delta) < \varepsilon/3$, then δ satisfies (3.21). When such a δ is chosen, denote $\eta := \eta(\delta)$. Denote $y_0 := x - \bar{x}$. Observe that $y_0 \in \operatorname{dom} \partial f$, hence $p_0 \in \partial f(y_0)$. Starting at $y_0' = y_0$ ESM generates a finite sequence $p_{i+1} \in \partial_\delta f(y_{i+1}')$, $i = 0, \ldots, n-2$. By the weaker version of the δ-subdifferential sum rule we have that

$$\partial_\delta f(\cdot) \subset \partial_\delta g(\cdot + \overline{x}) + \partial_\delta \langle -\bar{p}, \cdot \rangle + \partial_\delta 2c\| \cdot \|,$$

therefore,

$$p_{i+1} = q_{i+1}' - \overline{p}_{i+1} + \xi_{i+1}, \tag{3.29}$$

for some $q_{i+1}' \in \partial_\delta g(\cdot + \bar{x})(y_{i+1}')$, $\xi_{i+1} \in \partial_\delta 2c\| \cdot \|(y_{i+1}')$, and $\overline{p}_{i+1}$ such that $\|\overline{p}_{i+1} - \overline{p}\| \le \delta$, $i = 0, \ldots, n-2$. From (3.19) we have that

$$\langle p_{i+1}, y_{i+1}' - y_i' \rangle \le f(y_{i+1}') - f(y_i') + \varepsilon + \delta, \quad i = 0, \ldots, n-2.$$

Summing these equalities and using that $\delta < \eta$, we get

$$\sum_{i=0}^{n-2} \langle p_{i+1}, y'_{i+1} - y'_i \rangle \le f(y'_{n-1}) - f(y_0) + (n-1)(\varepsilon + \eta),$$

and from (3.29) we obtain that

$$\begin{aligned} \sum_{i=0}^{n-2} \langle q'_{i+1}, y'_{i+1} - y'_i \rangle &\le \sum_{i=0}^{n-2} \langle \overline{p}_{i+1}, y'_{i+1} - y'_i \rangle + \sum_{i=0}^{n-2} \langle \xi_{i+1}, y'_i - y'_{i+1} \rangle \\ &+ f(y'_{n-1}) - f(y_0) + (n-1)(\varepsilon + \eta). \end{aligned} \tag{3.30}$$

To estimate the right hand side of (3.30) we use, first, that

$$\begin{aligned} \sum_{i=0}^{n-2} \langle \overline{p}_{i+1}, y'_{i+1} - y'_i \rangle &\le \langle \overline{p}, y'_{n-1} - y_0 \rangle + \delta \sum_{i=0}^{n-2} \| y'_{i+1} - y'_i \| \\ &\le \langle \overline{p}, y'_{n-1} - y_0 \rangle + 2\delta \frac{f(x_0)}{c} \\ &\le \langle \overline{p}, y'_{n-1} - y_0 \rangle + \eta, \end{aligned}$$

second, that $\xi_{i+1} \in \partial_\delta 2c\| \cdot \|(y'_{i+1})$, hence

$$\begin{aligned} \sum_{i=0}^{n-2} \langle \xi_{i+1}, y'_i - y'_{i+1} \rangle &\le \sum_{i=0}^{n-2} \left(2c\|y'_i\| - 2c\|y'_{i+1}\| + \delta \right) \\ &= 2c\|y_0\| - 2c\|y'_{n-1}\| + (n-1)\delta \le 2c\|y_0\| + (n-1)\eta, \end{aligned}$$

and, third, that y'_{n-1} is an ε-minimum of f, hence $f(y'_{n-1}) \le \varepsilon$. Incorporating all these in (3.30) we obtain that

$$\sum_{i=0}^{n-2} \langle q'_{i+1}, y'_{i+1} - y'_i \rangle \le \langle \bar{p}, y'_{n-1} - y_0 \rangle + 2c\|y_0\| - f(y_0) + \tag{3.31}$$

$$(n-1)(\varepsilon + 2\eta) + \varepsilon + \eta.$$

By Brøndsted-Rockafellar Theorem there exist $q_{i+1} \in \partial g(\overline{x} + y_{i+1})$ such that $\|q_{i+1} - q'_{i+1}\| \le \sqrt{\delta}$, and $\|y_{i+1} - y'_{i+1}\| \le \sqrt{\delta}$. Then

$$\langle q_{i+1}, y_{i+1} - y_i \rangle - \langle q'_{i+1}, y'_{i+1} - y'_i \rangle =$$

$$\langle q_{i+1} - q'_{i+1}, y_{i+1} - y_i \rangle + \langle q'_{i+1}, y_{i+1} - y_i - y'_{i+1} + y'_i \rangle \le$$
$$\|q_{i+1} - q'_{i+1}\| \|y_{i+1} - y_i\| + \|q'_{i+1}\| (\|y_{i+1} - y'_{i+1}\| + \|y_i - y'_i\|).$$

Since $\|p_{i+1}\| \leq \|p_1'\|$, $\forall i$, which follows from (3.20), and since $\|p_1\| \leq \|p_0\|$ from (3.15), we easily derive that $\|q_{i+1}'\| \leq 2\delta + 2c + \|\overline{p}\| + \|p_0\|$, $\forall i$. Using the latter and $\|y_{i+1} - y_i\| \leq 2\sqrt{\delta} + \|y_{i+1}' - y_i'\|$ we obtain that

$$\langle q_{i+1}, y_{i+1} - y_i\rangle - \langle q_{i+1}', y_{i+1}' - y_i'\rangle \leq$$

$$\sqrt{\delta}(2\sqrt{\delta} + \|y_{i+1}' - y_i'\|) + 2\sqrt{\delta}(2\delta + 2c + \|\overline{p}\| + \|p_0\|) \leq \eta + \sqrt{\delta}\|y_{i+1}' - y_i'\|.$$

Hence,

$$\sum_{i=0}^{n-2}\langle q_{i+1}, y_{i+1} - y_i\rangle - \sum_{i=0}^{n-2}\langle q_{i+1}', y_{i+1}' - y_i'\rangle \leq$$

$$(n-1)\eta + \sqrt{\delta}\sum_{i=0}^{n-2}\|y_{i+1}' - y_i'\| \leq (n-1)\eta + 2\sqrt{\delta}\frac{f(x_0)}{c} \leq (n-1)\eta + \eta.$$

Using the latter in (3.31), as well as $\sqrt{\delta}\|\overline{p}\| \leq \eta$, and $\eta \leq \varepsilon/3$, we get

$$\begin{aligned}\sum_{i=0}^{n-2}\langle q_{i+1}, y_{i+1} - y_i\rangle &\leq \langle \bar{p}, y_{n-1} - y_0\rangle + 2c\|y_0\| - f(y_0)\\ &\quad + (n-1)(\varepsilon + 3\eta) + \varepsilon + 2\eta + \sqrt{\delta}\|\overline{p}\|\\ &\leq \langle \bar{p}, y_{n-1} - y_0\rangle + 2c\|y_0\| - f(y_0) + 2n\varepsilon. \end{aligned} \tag{3.32}$$

But

$$f(y_0) = f(x - \bar{x}) = g(x) - \langle \bar{p}, y_0\rangle - g(\bar{x}) + 2c\|y_0\|,$$

see (3.28), which combined with (3.32) yields

$$\sum_{i=0}^{n-2}\langle q_{i+1}, y_{i+1} - y_i\rangle \leq \langle \bar{p}, y_{n-1}\rangle + g(\overline{x}) - g(x) + 2n\varepsilon. \tag{3.33}$$

Now, let us denote $x_{i+1} := y_{i+1} + \bar{x}$, $i = 0, \ldots, n-2$. Then $q_{i+1} \in \partial g(x_{i+1})$, and $x_i - x_{i+1} = y_i - y_{i+1}$, $i = 0, \ldots, n-2$.

Setting $x_n = \bar{x}$ $y_n = 0$, and $q_n = \bar{p}$ from (3.33) we obtain that

$$\begin{aligned}
g(x)-g(\bar{x})-\sum_{i=0}^{n-1}\langle q_{i+1}, x_i - x_{i+1}\rangle &= g(x)-g(\bar{x})-\sum_{i=0}^{n-1}\langle q_{i+1}, y_i - y_{i+1}\rangle \\
&\leq \langle q_n, y_n - y_{n-1}\rangle + \langle \bar{p}, y_{n-1}\rangle + 2n\varepsilon \\
&= 2n\varepsilon \quad (\text{since } y_n = 0 \text{ and } q_n = \bar{p}) \\
&\leq 4\left(\sqrt{\frac{f(x_0)(\|p_0\|+1)}{c\varepsilon}}+1\right)\varepsilon \quad (\text{by } (3.23)) \\
&\leq 4\left(\sqrt{\frac{f(x_0)(\|p_0\|+1)}{c}}+1\right)\sqrt{\varepsilon} = M\sqrt{\varepsilon} \\
&< \varepsilon',
\end{aligned}$$

and (3.27) follows.

So far we have shown that $g(x) = g(\bar{x}) + R_{\partial g,(\bar{x},\bar{p})}(x)$ for $x \in \operatorname{dom}\partial g$. Now, fix any $x \in X$ and a real number r such that $r < g(x)$. By Proposition 3.1.4 we can find $(y,p) \in \operatorname{gph}\partial g$ such that

$$r < g(y) + \langle p, x - y\rangle.$$

Since $y \in \operatorname{dom}\partial g$ for a fixed $\varepsilon > 0$ we find a sequence $\{(x_i, q_i)\}_{i=1}^{n-1} \in \operatorname{gph}\partial g$ with $x_0 = y$, $x_n = \bar{x}$ and $q_n = \bar{p}$ such that

$$g(y) - g(\bar{x}) - \sum_{i=1}^{n-1}\langle q_{i+1}, x_i - x_{i+1}\rangle < \varepsilon.$$

Then,

$$r < g(\overline{x}) + \langle p, x - y\rangle + \sum_{i=1}^{n-1}\langle q_{i+1}, x_i - x_{i+1}\rangle + \varepsilon = g(\overline{x}) + \sum_{i=0}^{n-1}\langle q_{i+1}, x_i - x_{i+1}\rangle + \varepsilon,$$

where $q_1 := p$. Since $r < g(x)$ and $\varepsilon > 0$ were arbitrary, the proof is complete. □

Conclusion

Main contributions

In Chapter 1 we give a new proof of an intrinsic characterization of prox-regular sets in Hilbert spaces. Our prove avoids the use of properties of weakly convex sets and relies only on methods of classical analysis.
In Chapter 2 we provide a characterization of uniformly lower regular functions defined on a Hilbert space. To this end we introduce and study a property of epi prox-regularity of the epigraph set which slightly differs from the well known prox-regularity property of a set. This characterization is not a subdifferential one, but uses the properties of the epigraph.
In Chapter 3 we develop a novel variant of the classical epsilon subdifferential method in which the epsilon is fixed. We use our method to give a new prove of *Moreau-Rockafellar theorem* which states that a proper, lower semicontinuous and convex function defined on a Banach space is determined up to a constant by its subdifferential.

Publications related to the thesis

- M. Konstantinov and N. Zlateva, *Epsilon subdifferential method and integrability*, Journal of Convex Analysis **29** (2021), 571–582.

- M. Konstantinov, N. Zlateva, *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, Journal of Applied Analysis (2023)(to appear).

- M. Konstantinov, N. Zlateva, *Epigraphical characterization of uniformly lower regular functions in Hilbert spaces*, Journal of Convex Analysis (2023)(to appear).

Approbation

Some of the results contained in the thesis have been presented by the author at

the following conferences:

- *Epsilon Subdifferential Method And Integrability*, 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics, June 28–July 2, 2021, Borovets, Bulgaria, http://www.math.bas.bg/~bio/WP21/;
- *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", 26 March, 2022, Sofia, Bulgaria, https://www.fmi.uni-sofia.bg/bg/proletna-nauchna-sesiya-na-fmi-2022/;
- *Epsilon Subdifferential Method and Integrability*, 10-th International Conference on Numerical Methods and Applications, August 22–26, 2022, Borovets, Bulgaria, http://www.math.bas.bg/~nummeth/nma22/index.html;

and at the following seminars:

- *Epsilon Subdifferential Method and Integrability*, Optimization seminar, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", May 17, 2021, Sofia, Bulgaria;
- *Epsilon Subdifferential Method and Integrability*, Approximations, numerical methods and applications, September 26–29, 2021, Strelcha, Bulgaria;
- *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, Optimization seminar, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", March 21, 2021, Sofia, Bulgaria;
- *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, Numerical methods and applications, October 23–27, 2022, Strelcha, Bulgaria;
- *Epigraphical characterization of uniformly lower regular functions in Hilbert spaces*, Optimization seminar, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", December 12, 2022, Sofia, Bulgaria.

Declaration of originality

The author declares that the thesis contains original results obtained by him or in cooperation with his PhD supervisor. The results of other scientists were properly cited.

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Appendix

All notations come from the body of the thesis.

Proposition A.1.1. *Let C be a nonempty subset of H.Then $p \in N_C(x)$ if and only if there is a real $\sigma > 0$ such that*

$$\langle p, x' - x\rangle \leq \sigma \|x' - x\|^2, \quad \textit{for all } x' \in C.$$

Proof. Let $p \in N_C(x)$, i.e. there exist $r > 0$ such that $x \in P_C(x+r\,p)$. The latter equivalent to

$$\|rp\| = \|x + rp - x\| = d_C(x + rp),$$

which is equivalent to

$$\|rp\|^2 \leq \|x + rp - x'\|^2, \quad \text{for all } x' \in C,$$

which is equivalent to

$$\|rp\|^2 \leq \langle x - x' + rp, x - x' + rp\rangle, \quad \text{for all } x' \in C,$$

which is equivalent to

$$\|rp\|^2 \leq \|x' - x\|^2 + 2\langle x - x', rp\rangle + \|rp\|^2, \quad \text{for all } x' \in C,$$

which is finally equivalent to

$$\langle p, x' - x\rangle \leq \frac{1}{2r}\|x' - x\|^2, \quad \text{for all } x' \in C.$$

and for $\sigma = \frac{1}{2r}$ the proof is complete. □

Corollary A.1.2. *For any $x \in C$ the set $N_C(x)$ is a cone.*

Proof. Let $x \in C$, $p \in N_C(x)$ and $t \geq 0$.

We will show that $tp \in N_C(x)$.

If $t = 0$, then it is clear that $tp = 0 \in N_C(x)$, since $x \in P_C(x)$.

Assume that $t > 0$. Since $p \in N_C(x)$ from Proposition A.1.1 we have that

$$\langle p, x' - x \rangle \leq \frac{1}{2r}\|x' - x\|^2, \quad \text{for all } x' \in C.$$

Hence

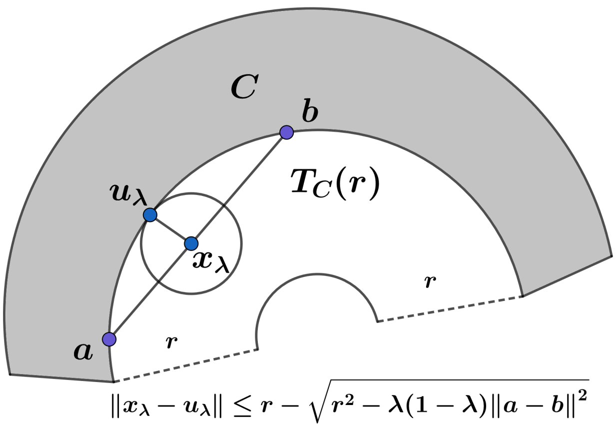
$$\langle tp, x' - x \rangle \leq \frac{1}{2r_t}\|x' - x\|^2, \quad \text{for all } x' \in C,$$

where

$$r_t := \frac{r}{t} > 0.$$

The latter and Proposition A.1.1 yield that $tp \in N_C(x)$. The proof is complete. $\square$

Proposition A.1.3. *The statement of Remark 1.2.1 holds, i.e. for any $\lambda \in (0, 1)$ we have that $u_\lambda \notin \{a, b\}$.*



Proof. Suppose $u_\lambda = a$.

Hence,

$$\|x_\lambda - a\| \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2},$$

or

$$(1 - \lambda)\|a - b\| \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2},$$

or

$$\sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2} \leq r - (1 - \lambda)\|a - b\|. \tag{A.1}$$

Recall that $a = u_\lambda = p_C(x_\lambda)$, which means that

$$\|x - a\| \leq \|x - x'\|, \quad \text{for all } x' \in C.$$

Substituting $x' = b$ in the latter gives

$$(1-\lambda)\|a-b\| \leq \lambda\|a-b\|,$$

which yields that

$$(1-\lambda) \leq \frac{1}{2}.$$

From the latter and since $\|a-b\| < 2r$ the right-hands side of inequality (A.1) is nonnegative. Hence raising (A.1) to the power of two gives

$$r^2 - \lambda(1-\lambda)\|a-b\|^2 \leq r^2 - 2r(1-\lambda)\|a-b\| + (1-\lambda)^2\|a-b\|^2,$$

or

$$2r(1-\lambda)\|a-b\| \leq \big(\lambda(1-\lambda) + (1-\lambda)^2\big)\|a-b\|^2,$$

or

$$2r(1-\lambda)\|a-b\| \leq (1-\lambda)\|a-b\|^2,$$

or finally

$$2r \leq \|a-b\|,$$

which is a contradiction with $\|a-b\| < 2r$. The case $u_\lambda = b$ can be proven analogously. □

Proposition A.1.4. *For a fixed $t \geq 0$ the function*

$$f : [\sqrt{t}, \infty) \to \mathbb{R}^+$$

defined as

$$f(r) := r - \sqrt{r^2 - t},$$

is convex decreasing and for any $r \geq \sqrt{t}$ satisfies

$$f(r) \geq \frac{t}{2r}.$$

Proof. For $r > \sqrt{t}$ we have that

$$f'(r) = 1 - \frac{r}{\sqrt{r^2 - t}} = \frac{\sqrt{r^2 - t} - r}{\sqrt{r^2 - t}} < 0,$$

and

$$f''(r) = \frac{t}{(r^2 - t)^{\frac{3}{2}}} > 0.$$

Using classical results in Calculus we have that since $f'(r) < 0$ and $f''(r) > 0$ the function f is convex decreasing. Also since $\frac{t^2}{4r^2} \geq 0$ we have that

$$r^2 - t + \frac{t^2}{4r^2} \geq r^2 - t,$$

or equivalently

$$\left(r - \frac{t}{2r}\right)^2 \geq r^2 - t.$$

Note that the right side of the latter is nonnegative by the choice of r. Hance by taking the square root we get

$$r - \frac{t}{2r} \geq \sqrt{r^2 - t},$$

or equivalently

$$r - \sqrt{r^2 - t} \geq \frac{t}{2r}.$$

□

Proposition A.1.5. *The distance function d_C is Lipschitz continuous with constant 1, i.e.*

$$|d_C(x) - d_C(y)| \leq \|x - y\|, \text{ for all } x, y \in H.$$

Proof. Let $x, y \in C$ and $\varepsilon > 0$ be fixed. From the definition of the distance function there exist $z_0 \in C$ such that

$$\|y - z_0\| < d_C(y) + \varepsilon,$$

and

$$\begin{aligned} d_C(x) &= \inf_{u \in C} \|x - u\| \leq \|x - z_0\| \\ &\leq \|x - y\| + \|y - z_0\| \leq |x - y\| + d_C(y) + \varepsilon. \end{aligned}$$

Hence,

$$d_C(x) - d_C(y) < \|x - y\| + \varepsilon.$$

Letting ε tend to zero in the latter we achieve

$$d_C(x) - d_C(y) \leq \|x - y\|,$$

and by replacing x and y and repeating the same reasoning we get

$$d_C(y) - d_C(x) \leq \|y - x\|.$$

Hence

$$\max\left(d_C(x) - d_C(y), d_C(y) - d_C(x)\right) = |d_C(x) - d_C(y)| \leq \|x - y\|.$$

The proof is complete. □

Proposition A.1.6. *The following inclusion holds*

$$N_C(x) \subseteq N_C^F(x), \text{ for all } x \in C.$$

Proof. Let $x \in C$ and $x^* \in N_C(x)$. From Proposition A.1.1 there exist $r > 0$ such that

$$\langle x^*, x' - x \rangle \leq \frac{1}{2r} \|x' - x\|^2, \text{ for all } x' \in C.$$

Let us fix $\varepsilon > 0$ and denote $U := B(x, 2r\varepsilon)$. Then for any $x' \in C \cap U$ we have that

$$\langle x^*, x' - x \rangle \leq \frac{1}{2r} \|x' - x\| \|x' - x\| \leq \frac{1}{2r} 2r\varepsilon \|x' - x\| = \varepsilon \|x' - x\|.$$

Hence $x^* \in N_C^F(x)$. The proof is complete.

Note: One can prove the latter considering that the norm in H is Fréchet differentiable away from the origin (see [17, Corollary 3.1]). □

Proposition A.1.7. *If we have that for* $(x, \alpha) \in \operatorname{epi} f$

$$(p, \beta) \in N_{\operatorname{epi} f}(x, \alpha),$$

then

$$\beta \leq 0.$$

Proof. Assume that $\beta > 0$. Since $(p, \beta) \in N_{\operatorname{epi} f}(x, \alpha)$ there exists $r > 0$ such that

$$\big\langle (p, \beta), (x', \alpha') - (x, \alpha) \big\rangle \leq \frac{1}{2r} |||(x', \alpha')' - (x, \alpha)|||^2, \tag{A.2}$$

$$\text{for all } (x', \alpha') \in N_{\operatorname{epi} f}(x, \alpha).$$

Since $(x, \alpha) \in \operatorname{epi} f$ we have $f(x) \leq \alpha$. Let ε be such that

$$0 < \varepsilon < 2r\beta.$$

We have that

$$f(x) \leq \alpha < \alpha + \varepsilon,$$

hence

$$(x, \alpha + \varepsilon) \in \operatorname{epi} f.$$

So if we substitute $(x', \alpha') = (x, \alpha + \varepsilon)$ in (A.2) we get

$$\big\langle (p, \beta), (x, \alpha + \varepsilon) - (x, \alpha) \big\rangle \leq \frac{1}{2r} |||(x, \alpha + \varepsilon) - (x, \alpha)|||^2,$$

or

$$\big\langle (p, \beta), (0, \varepsilon) \big\rangle \leq \frac{1}{2r} |||(0, \varepsilon)|||^2,$$

The latter yields the following contradiction

$$\beta\varepsilon \leq \frac{1}{2r}\varepsilon^2,$$

or by the choice of ε

$$\beta \leq \frac{\varepsilon}{2r} < \frac{2r\beta}{2r} = \beta.$$

Hence $\beta \leq 0$. □

Proposition A.1.8. *If $(x, \alpha) \in \operatorname{epi} f$ and $(x^*, -\lambda) \in N_{\operatorname{epi} f}(x, \alpha)$ for $\lambda > 0$, then*

$$f(x) = \alpha.$$

Proof. Since $(x, \alpha) \in \operatorname{epi} f$ we have that $f(x) \leq \alpha$. Let us assume that $f(x) < \alpha$. From Proposition A.1.1 there exist $r > 0$ such that

$$\big\langle (x^*, -\lambda), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|(x' - x, \alpha' - \alpha)\|^2, \tag{A.3}$$

$$\text{for all } (x', \alpha') \in \operatorname{epi} f.$$

Let γ be such that

$$0 < \gamma < \min\left(2r\lambda, \alpha - f(x)\right).$$

From the latter we have that $f(x) < \alpha - \gamma$, hence $(x, \alpha - \gamma) \in \operatorname{epi} f$. Substituting $(x', \alpha') = \big(x, \alpha - \gamma\big)$ in (A.3) we get that

$$\big\langle (x^*, -\lambda), (0, \alpha - \gamma - \alpha) \big\rangle \leq \frac{1}{2r} \|(x^*, -\lambda), (0, \alpha - \gamma - \alpha)\|^2,$$

or

$$\lambda\gamma \leq \frac{1}{2r}\gamma^2,$$

which from the choice of γ gives the following contradiction

$$2r\lambda \leq \gamma < 2r\lambda.$$

Hence $f(x) = \alpha$. The proof is complete. □

Proposition A.1.9. *Let $x \in C \subset \overline{H}$ and $v \in N_C(x)$. Then one has*

$$(v, -1) \in N_{\operatorname{epi} d_C}\big(x, d_C(x)\big).$$

Proof. From (1.3) we have that there exist $\sigma > 0$ such that

$$\langle v, x' - x\rangle \leq \sigma |||x' - x|||^2, \text{ for all } x' \in C.$$

Since $d_C(x) = 0$, we have that for all $(x', \alpha') \in \operatorname{epi} d_C$

$$\big\langle (v, -1), (x', \alpha') - (x, d_C(x))\big\rangle = \langle v, x' - x\rangle - \alpha' \leq \langle v, x' - x\rangle$$

The last inequality above holds because $\alpha' \geq d_C(x') \geq 0$. Hence we have that

$$\big\langle (v, -1), (x', \alpha') - (x, d_C(x))\big\rangle \leq \langle v, x' - x\rangle \leq \sigma |||x' - x|||^2 \leq$$
$$\leq \sigma \left(\|x' - x\|^2 + (\alpha' - d_C(x))^2\right) = \sigma ||||(x', \alpha') - (x, d_C(x))||||^2,$$
$$\text{for all } (x', \alpha') \in \operatorname{epi} d_C.$$

The proof is complete. □

Proposition A.1.10. *The function*

$$\varphi_{x_0}(K) := \inf_{x\in\mathbb{R}^n} F_{x_0}(K, x),$$

where

$$F_{x_0}(K, x) := f(x) - f(x_0) + \varepsilon + K\|x - x_0\|,$$

is locally Lipschitz in $(0, \infty)$.

Proof. Let $K > 0$ be fixed and K', K'' be such that

$$\text{(A.4)} \qquad |K' - K| < \frac{K}{2} \quad \text{and} \quad |K'' - K| < \frac{K}{2}.$$

Let $\delta > 0$ be fixed and $x'_\delta \in \mathbb{R}^n$ be a δ-minimum point for the function

$$F_{x_0}(\,\cdot\,, K') = f(\cdot) - f(x_0) + \varepsilon + K'\|\cdot - x_0\|,$$

i.e. for all $x \in \mathbb{R}^n$,

$$\text{(A.5)} \qquad f(x'_\delta) - f(x_0) + \varepsilon + K'\|x'_\delta - x_0\| \leq f(x) - f(x_0) + \varepsilon + K'\|x - x_0\| + \delta.$$

After substituting $x = x_0$ in (A.5) we get

$$\begin{aligned} f(x_{\delta'}) - f(x_0) + \varepsilon + K'\|x\delta' - x_0\| &\leq f(x_0) - f(x_0) + \varepsilon + K'\|x_0 - x_0\| + \delta \\ &= \varepsilon + \delta, \end{aligned}$$

or

$$\text{(A.6)} \qquad f(x_{\delta'}) - f(x_0) + K'\|x\delta' - x_0\| \leq \delta.$$

Denoting $m := \dfrac{2\,(\,f(x_0)+\delta)}{K} > 0$ and using (A.6) we have the following chain of inequalities

$$\begin{aligned}\|x'_\delta - x_0\| &\le \frac{f(x_0)-f(x'_\delta)+\delta}{K'} && \text{since } K' > \frac{K}{2} > 0\\ &\le \frac{f(x_0)+\delta}{K'} && \text{since } f(x'_\delta) \ge 0\\ &\le \frac{2(f(x_0)+\delta)}{K} = m && \text{since } \frac{1}{K'} < \frac{2}{K}.\end{aligned}$$

Finally,

$$\|x'_\delta - x_0\| \le m. \tag{A.7}$$

Analogously, for $x''_\delta \in \mathbb{R}^n$ which is a δ-minimum point for the function

$$F_{x_0}(\,\cdot\,, K'') = f(\cdot) - f(x_0) + \varepsilon + K''\|\cdot - x_0\|,$$

we obtain that

$$\|x''_\delta - x_0\| \le m. \tag{A.8}$$

Now, since $\varphi_{x_0}(K') := \inf_{x\in\mathbb{R}^n} F_{x_0}(K', x) \le f(x''_\delta) - f(x_0) + \varepsilon + K'\|x''_\delta - x_0\|$ and x''_δ is δ-minimum of $F_{x_0}(\,\cdot\,, K'')$ we have that

$$\begin{aligned}\varphi_{x_0}(K') - \varphi_{x_0}(K'') &\le f(x''_\delta) - f(x_0) + \varepsilon + K'\|x''_\delta - x_0\|\\ &\quad -f(x''_\delta) + f(x_0) - \varepsilon - K''\|x''_\delta - x_0\| + \delta\\ &= (K' - K'')\,\|x''_\delta - x_0\| + \delta\\ &\le |K' - K''|\|x''_\delta - x_0\| + \delta\\ &\le m|K' - K''| + \delta \qquad \text{by (A.8)},\end{aligned}$$

or

$$\varphi_{x_0}(K') - \varphi_{x_0}(K'') \le m|K' - K''| + \delta. \tag{A.9}$$

Analogously, using x'_δ instead of x''_δ, we get that

$$\varphi_{x_0}(K'') - \varphi_{x_0}(K') \le m|K' - K''| + \delta. \tag{A.10}$$

Combining (A.9) and (A.10) and letting δ tend to zero we obtain that for all K', K'' satisfying (A.4),

$$|\varphi_{x_0}(K') - \varphi_{x_0}(K'')| \le m|K' - K''| \tag{A.11}$$

which yields that φ_{x_0} is locally Lipschitz on $(0,\infty)$. □

Proposition A.1.11. *Assume that* $0 \notin \partial_\varepsilon f(x_0)$. *Then for the function* $\varphi_{x_0}(K)$ *there exist* $\widetilde{K}_0, \widehat{K}_0 > 0$ *such that* $\varphi_{x_0}(\widetilde{K}_0) < 0$ *and* $\varphi_{x_0}(\widehat{K}_0) > 0$.

Proof. Since $0 \notin \partial_\varepsilon f(x_0)$ there exist $x' \in \mathbb{R}^n$ such that $f(x') < f(x_0) - \varepsilon$.
Let $\delta > 0$ be such that

$$f(x') = f(x_0) - \varepsilon - \delta. \tag{A.12}$$

Take $\widetilde{K}_0 \in \mathbb{R}$ satisfying

$$0 < \widetilde{K}_0 < \frac{\delta}{\|x' - x_0\|}. \tag{A.13}$$

The following chain of equalities and inequalities that $\varphi_{x_0}(\widetilde{K}_0) < 0$.

$$\begin{aligned}
\varphi_{x_0}(\widetilde{K}_0) &= \inf_{x \in \mathbb{R}^n} \Big(f(x) - f(x_0) + \varepsilon + \widetilde{K}_0 \|x - x_0\| \Big) \\
&\leq f(x') - f(x_0) + \varepsilon + \widetilde{K}_0 \|x' - x_0\| \\
&= f(x_0) - \varepsilon - \delta - f(x_0) + \varepsilon + \widetilde{K}_0 \|x' - x_0\| \text{ from (A.12)} \\
&= \widetilde{K}_0 \|x' - x_0\| - \delta \\
&< \frac{\delta}{\|x' - x_0\|} \|x' - x_0\| - \delta \quad \text{from (A.13)} \\
&= 0.
\end{aligned}$$

To show that there exist $\widehat{K}_0 > 0$ such that $\varphi_{x_0}(\widehat{K}_0) > 0$, assume that contrary, i.e. that for any $K > 0$ there exists x_k such that

$$f(x_k) - f(x_0) + \varepsilon + K\|x_k - x_0\| \leq 0,$$

i.e.

$$\|x_k - x_0\| \leq \frac{f(x_0) - f(x_k) - \varepsilon}{K}. \tag{A.14}$$

Note that since φ_{x_0} is strictly monotone increasing on $(0, \infty)$ if for some $K' > 0$ we have $\varphi_{x_0}(K') = 0$ then for some $K'' > K'$ we will have that $\varphi_{x_0}(K'') > 0$ which will contradict with the assumtions. Hence can assume that for all $K > 0$ we have that $\varphi_{x_0}(K) < 0$ which gives (A.14).

Now take $\delta \in \mathbb{R}$ such that

$$0 < \delta < \varepsilon. \tag{A.15}$$

Since f is lower semicontinuous at x_0 there exists $\widetilde{\delta} > 0$ such that for all x such that $\|x - x_0\| < \widetilde{\delta}$ it holds that

$$f(x_0) \leq f(x) + \delta. \tag{A.16}$$

Let $\overline{K} > 0$ be large enough that

$$\frac{f(x_0) - \varepsilon}{\overline{K}} < \widetilde{\delta}. \tag{A.17}$$

Let $x_{\overline{k}}$ correspond to $\overline{K} > 0$ in our assumption.
Using (A.14) and (A.17) it is easy to show that $\|x_{\overline{k}} - x_0\| \leq \widetilde{\delta}$.
Hence for $x_{\overline{k}}$ we have that

$$\begin{aligned} \|x_{\overline{k}} - x_0\| &\leq \frac{f(x_0) - f(x_{\overline{k}}) - \varepsilon}{\overline{K}} \quad \text{from (A.14)} \\ &\leq \frac{f(x_{\overline{k}}) + \delta - f(x_{\overline{k}}) - \varepsilon}{\overline{K}} \quad \text{from (A.16), (A.17) and } f(x_{\overline{k}}) \geq 0 \\ &= \frac{\delta - \varepsilon}{\overline{K}} < 0, \quad \text{from (A.15)} \end{aligned}$$

which yields to a contradiction. Hence, the proof is complete. □

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