Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces

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Abstract

We provide new proofs of two intrinsic properties of prox-regular sets in Hilbert spaces.

Key words: prox-regular set, uniformly prox-regular set, proximally smooth set, proximal normal, distance function, metric projection mapping, Hilbert space

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1 Introduction

The study of prox-regular sets, a term due to Poliquin, Rockafellar and Thibault [15], can be traced back to the pioneering work [12] of Federer who introduced them as positively reached sets in $\mathbb{R}^n$. During the years, various names of such sets have been introduced: weakly convex ([18]) or proximally smooth sets ([8]) are commonly used in Hilbert spaces; for other names see the survey [9]. Prox-regular sets in Banach spaces are studied in [6, 7, 4, 5].

Along with the study of prox-regular sets from theoretical point of view, they are intensively studied as involved in the famous Moreau's sweeping processes, see the survey [13] and the references therein. Various stability and separation properties of prox-regular sets are established in [1, 2, 3]. More details one can find in the paper [15], the survey [9], the forthcoming book [17] and their bibliography.

Prox-regularity has been introduced as an important new regularity property in Variational Analysis by Poliquin and Rockafellar in [14]. They defined the concept for functions and sets and developed the subject in $\mathbb{R}^n$. Numerous significant characterizations of prox-regularity of a closed set C in Hilbert space at point $\overline{x} \in C$ were obtained by Poliquin, Rockafellar and Thibault in [15] in terms of the distance function d_C and metric projection mapping P_C, e.g. d_C being continuously differentiable outside of C on a neighbourhood of $\overline{x}$, or P_C being single-valued and norm-to-weak continuous on this same neighbourhood. On global level, there the authors showed that uniformly prox-regular sets are proximally smooth sets provided new insights on them.

In this note we prove the following intrinsic characteristic properties of a r-prox-regular set.

Theorem 1.1. *Given a real $r > 0$, a non-empty closed set C in a Hilbert space H. The following are equivalent:*
(a) C is r-prox-regular.
(b) For any $a, b \in C$ with $\|a - b\| < 2r$ and any $\lambda \in (0, 1)$ for $x_\lambda := \lambda a + (1 - \lambda)b$ there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \leq r - \sqrt{r^2 - \lambda(1 - \lambda)\|a - b\|^2}. \tag{1.1}$$

(c) For any $a, b \in C$ with $\|a - b\| < 2r$ there is some $z \in C$ such that

$$\left\| \frac{a + b}{2} - z \right\| \leq r - \sqrt{r^2 - \frac{\|a - b\|^2}{4}}. \tag{1.2}$$

The equivalence (a) $\Leftrightarrow$ (c) is established by G. E. Ivanov, see [11, Lemma 4.2] by using the properties of the sets $\Delta_r(a,b) := \bigcap_{d:\{a,b\}\in B[d,r]}$, first considered by J.-P. Vial, see [18]. In our proof we use a different approach which does not rely on these sets.

In finite dimensional settings, J.-P. Vial, see [18, Proposition 3.4], proved the implication (a) $\Rightarrow$ (b) with right hand side of (1.1) equal to $\theta_\lambda := \dfrac{\lambda(1-\lambda)}{r}\|a-b\|^2$, and the implication (b) $\Rightarrow$ (a) with right hand side of (1.1) equal to $\delta_\lambda := \dfrac{\lambda(1-\lambda)}{2r}\|a-b\|^2$ As $\delta_\lambda < r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} < \theta_\lambda$, the condition (1.1) is slightly weaker than both conditions of Vial. The equivalence (a) $\Leftrightarrow$ (b) is proved in Hilbert settings in [17, Proposition 15.41], by using different arguments.

The characteristic properties (1.1) and (1.2) of r-prox-regular set will be studied in forthcoming paper of the authors, in the context of epigraphs of functions.

2 Preliminaries and notations

Throughout the paper, H stands for a (real) Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and with the associated with it norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. The open (resp. closed) ball and the sphere of H centered at $x \in H$ with radius $t > 0$ is denoted by $B(x,t)$ (resp. $B[x,t]$). In the particular case of the closed unit ball we use the notation $\mathbb{B} := B[0;1]$.

For any nonempty subset C of H the distance function d_C from C is defined as

$$d_C(x) := \inf_{y\in C} \|x-y\|, \qquad \text{for all } x \in H.$$

For an extended real $r \in (0,+\infty]$ through the distance function, one defines the (open) r-tube of C as the set $T_C(r) := U_C(r) \setminus C$, where $U_C(r)$ is the (open) r-enlargement of C

$$U_C(r) := \{x \in H : d_C(x) < r\}.$$

The multi-valued mapping $P_C : H \rightrightarrows H$ of nearest points in C is defined by

$$P_C(x) := \{y \in C : d_C(x) = \|x-y\|\} \quad \text{for all } x \in H.$$

Whenever for some $\overline{x} \in H$ the latter set is reduced to a singleton, i.e. $P_C(\overline{x}) = \{\overline{y}\}$, the vector $\overline{y} \in H$ is denoted by $p_C(\overline{x})$.

The proximal normal cone of C at $x \in H$, denoted by $N_C(x)$, is defined as, see [16],

$$N_C(x) := \{p \in H : \exists r > 0 \text{ such that } x \in P_C(x + rp)\}.$$

By convention, $N_C(x') = \varnothing$ for all $x' \notin C$. It is easy to see that $p \in N_C(x)$ if and only if there is a real $r > 0$ such that

$$\langle p, x' - x\rangle \leq \frac{1}{2r}\|x' - x\|, \quad \text{for all } x' \in C \tag{2.1}$$

in which case one says that p is a proximal normal to C at x with constant $r > 0$.

Definition 2.1. *Let C be a nonempty closed subset of H and $r \in (0, +\infty]$. One says that C is r-prox-regular (or uniformly prox-regular with constant r) whenever, for every $x \in C$, for every $p \in N_C(x) \cap \mathbb{B}$ and for every real $t \in (0, r]$, one has*

$$x \in P_C(x + tp).$$

Given a closed subset $C \in H$, $x \in C$ and $p \in N_C(x)$ with $\|p\| = 1$, it is known that for every real $t > 0$ one has

$$x \in P_C(x + tp) \Leftrightarrow C \cap B(x + tp, t) = \varnothing.$$

In such a case, one says that the unit normal proximal vector p to C at x is realized by the t-ball $B(x + tv, t)$.

In the following theorem are collected some of the characterizations of uniformly prox-regular sets for which we refer to [15].

Theorem 2.2. *Let C be a nonempty closed subset of H and let $r > 0$. The following assertions are equivalent:*
(a) The set C is r-prox-regular.
(b) For all $x, x' \in C$, for all $p \in N_C(x)$, one has

$$\langle p, x' - x\rangle \leq \frac{1}{2r}\|p\|\|x' - x\|. \tag{2.2}$$

(c) P_C is single-valued and norm-to-weak continuous on $T_C(r)$.

3 Proof of Theorem 1.1

The statements will be proved in the order (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let C be r-prox-regular. Let $a, b \in C$ with $\|a-b\| < 2r$ and $\lambda \in (0,1)$ be fixed, and let $x_\lambda = \lambda a + (1-\lambda)b$.

It is obvious that $x_\lambda \in U_C(r)$. If $x_\lambda \in C$, we just take $u_\lambda = x_\lambda$. Otherwise, $x_\lambda \in T_C(r)$. From Theorem 2.2(c) there exists unique $u_\lambda \in C$ such that $u_\lambda := p_C(x_\lambda)$.

Since λ is fixed, further we will omit it from the index and will work with $x := x_\lambda$, and $u := u_\lambda$ instead. Set $p := x - u$ and observe that $p \neq 0$ and that $p \in N_C(u)$. From Theorem 2.2(b) it holds that

$$\langle p, x' - u \rangle \leq \frac{1}{2r}\|p\|\|x' - u\|, \quad \forall x' \in C. \tag{3.1}$$

It is clear that

$$u = \lambda a + (1-\lambda)b - p. \tag{3.2}$$

Substituting $x' = a$ in (3.1) and using the expression (3.2) for u, we get

$$\langle p, (1-\lambda)(a-b) + p \rangle \leq \frac{1}{2r}\|p\|\|(1-\lambda)(a-b) + p\|^2 =$$

$$= \frac{1}{2r}\|p\| \left((1-\lambda)^2\|a-b\|^2 + 2(1-\lambda)\langle a-b, p\rangle + \|p\|^2 \right). \tag{3.3}$$

Analogously, substituting $x' = b$ in (3.1) we have

$$\langle p, \lambda(b-a) + p \rangle \leq \frac{1}{2r}\|p\| \left(\lambda^2\|b-a\|^2 + 2\lambda\langle b-a, p\rangle + \|p\|^2 \right). \tag{3.4}$$

Multiplying inequality (3.3) by λ, inequality (3.4) by $(1-\lambda)$ and adding them, we obtain

$$\langle p, p \rangle \leq \frac{1}{2r}\|p\| \left(\lambda(1-\lambda)\|a-b\|^2 + \|p\|^2 \right).$$

Rearranging the latter, we have that $\|p\|$ satisfies the following quadratic inequality

$$t^2 - 2rt + \lambda(1-\lambda)\|a-b\|^2 \geq 0. \tag{3.5}$$

Since $\|a-b\| < 2r$, and $\lambda \in (0,1)$,

$$D := 4r^2 - 4\lambda(1-\lambda)\|a-b\|^2 > 0$$

and any t satisfying (3.5) should satisfy $t \le t_1$ or $t \ge t_2$, where

$$t_1 := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}, \quad t_2 := r + \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Having in mind that $u = p_C(x)$, we have

$$\|p\| = \|x-u\| \le \|x-a\| = \|\lambda a + (1-\lambda)b - a\| = (1-\lambda)\|b-a\|,$$

and

$$\|p\| = \|x-u\| \le \|x-b\| = \|\lambda a + (1-\lambda)b - b\| = \lambda\|b-a\|.$$

Hence

$$\|p\| \le \frac{\|b-a\|}{2} < \frac{2r}{2} = r.$$

As $t_2 \ge r$, we obviously get that $\|p\| \le t_1$, which reads

$$\|p\| \le r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2},$$

and the proof of (a) $\Rightarrow$ (b) is completed. It is straightforward to check that u_λ for any $\lambda \in (0,1)$ is such that $u_\lambda \notin \{a,b\}$.

(b) $\Rightarrow$ (c) is obvious, just take $\lambda = \dfrac{1}{2}$ in (1.1).

(c) $\Rightarrow$ (a). Let x_0 be any point in $T_C(r)$, i.e. $0 < d_C(x_0) < r$. Set

$$\Delta := \frac{1}{2}\min\{d_C(x_0), r - d_C(x_0)\}.$$

Take arbitrary $x \in B(x_0, \Delta)$. Note that by the choice of Δ,

$$d_C(x) \le d_C(x_0) + \|x - x_0\| < r - 2\Delta + \Delta = r - \Delta$$

and

$$d_C(x) \ge d_C(x_0) - \|x - x_0\| > 2\Delta - \Delta = \Delta.$$

Setting $d := d_C(x)$, we have

$$\Delta < d < r - \Delta. \tag{3.6}$$

Take any $\varepsilon \in (0, \Delta)$.

Take $a, b \in C$, $a \ne b$ such that $a, b \in \varepsilon - \operatorname{argmin} d_C(x)$, and $\|a-b\| > \varepsilon$ (if any).

Since $\|a - x\| \le d + \varepsilon$, and $\|b - x\| \le d + \varepsilon$,

$$\|a - b\| \le \|a - x\| + \|b - x\| \le 2d + 2\varepsilon < 2(r - \Delta) + 2\Delta = 2r.$$

From (1.2) there exists $z \in C$ such that

$$\left\| \frac{a+b}{2} - z \right\| \le r - \sqrt{r^2 - \frac{\|a-b\|^2}{4}}. \tag{3.7}$$

Setting $\overline{a} := x + d\frac{a-x}{\|a-x\|}$ we have a point $\overline{a}$ such that $\|\overline{a} - x\| = d$ and $\|\overline{a} - a\| \le \varepsilon$. Analogously, we obtain a point $\overline{b}$ such that $\|\overline{b} - x\| = d$ and $\|\overline{b} - b\| \le \varepsilon$. Moreover, $\overline{a} \neq \overline{b}$ (otherwise one gets a contradiction with $\|a - b\| > \varepsilon$.)

Since

$$\begin{aligned}\left\| \frac{\overline{a}+\overline{b}}{2} - x \right\|^2 &= 2\left\| \frac{\overline{a}-x}{2} \right\|^2 + 2\left\| \frac{\overline{b}-x}{2} \right\|^2 - \left\| \frac{\overline{a}-\overline{b}}{2} \right\|^2 \\ &= \frac{1}{2}d^2 + \frac{1}{2}d^2 - \frac{\|\overline{a}-\overline{b}\|^2}{4} = d^2 - \frac{\|\overline{a}-\overline{b}\|^2}{4},\end{aligned}$$

one has

$$\left\| \frac{\overline{a}+\overline{b}}{2} - x \right\| = \sqrt{d^2 - \frac{1}{4}\|\overline{a}-\overline{b}\|^2}. \tag{3.8}$$

Since $d_C(x) = d$, any ball centered at $\frac{\overline{a}+\overline{b}}{2}$ with radius smaller than $d - \sqrt{d^2 - \frac{1}{4}\|\overline{a}-\overline{b}\|^2}$ does not contain any point of the set C. But $z \in C$, hence it holds that

$$\left\| \frac{\overline{a}+\overline{b}}{2} - z \right\| \ge d - \sqrt{d^2 - \frac{1}{4}\|\overline{a}-\overline{b}\|^2}. \tag{3.9}$$

Combining (3.9) with (3.7), we get

$$\begin{aligned} d-\sqrt{d^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2} &\leq \left\|\frac{\overline{a}+\overline{b}}{2}-z\right\| \\ &\leq \left\|\frac{a+b}{2}-z\right\| + \frac{1}{2}\|\overline{a}-a+\overline{b}-b\| \\ &\leq \left\|\frac{a+b}{2}-z\right\| + \varepsilon \\ &\leq r-\sqrt{r^2-\frac{1}{4}\|a-b\|^2}+\varepsilon. \end{aligned} \tag{3.10}$$

Since $\|a-b\| \leq \|\overline{a}-\overline{b}\|+2\varepsilon$, it holds that $\dfrac{\|a-b\|^2}{4} \leq \dfrac{(\|\overline{a}-\overline{b}\|+2\varepsilon)^2}{4}$ and

$$r^2-\frac{\|a-b\|^2}{4} \geq r^2-\frac{(\|\overline{a}-\overline{b}\|+2\varepsilon)^2}{4} > 0. \tag{3.11}$$

where the strict inequality holds since by (3.6) and $\varepsilon < \Delta$ we have

$$\|\overline{a}-\overline{b}\|+2\varepsilon \leq 2d+2\varepsilon < 2(r-\Delta)+2\Delta = 2r.$$

From (3.11) we get that

$$r-\sqrt{r^2-\frac{\|a-b\|^2}{4}} \leq r-\sqrt{r^2-\frac{(\|\overline{a}-\overline{b}\|+2\varepsilon)^2}{4}}$$

which combined with (3.10) gives

$$d-\sqrt{d^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2} \leq r-\sqrt{r^2-\frac{1}{4}(\|\overline{a}-\overline{b}\|+2\varepsilon)^2}+\varepsilon =$$

$$r-\sqrt{r^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}+\sqrt{r^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}-\sqrt{r^2-\frac{1}{4}(\|\overline{a}-\overline{b}\|+2\varepsilon)^2}+\varepsilon.$$

It is straightforward to obtain the estimation

$$\sqrt{r^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}-\sqrt{r^2-\frac{1}{4}(\|\overline{a}-\overline{b}\|+2\varepsilon)^2} \leq \left(2\sqrt{\frac{r}{\Delta}}\right)\varepsilon.$$

Hence,

$$d-\sqrt{d^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2} \leq r-\sqrt{r^2-\frac{1}{4}\|\overline{a}-\overline{b}\|^2}+\left(2\sqrt{\frac{r}{\Delta}}+1\right)\varepsilon,$$

and setting $c(\Delta) := \left(2\sqrt{\frac{r}{\Delta}} + 1\right)$ we have

$$d - \sqrt{d^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} \leq r - \sqrt{r^2 - \frac{1}{4}\|\overline{a} - \overline{b}\|^2} + c(\Delta)\varepsilon. \tag{3.12}$$

For fixed $t > 0$ consider the function $f : [t, \infty) \to \mathbb{R}$ defined as

$$f(r) := r - \sqrt{r^2 - t}.$$

It is easy to check that it is a convex decreasing function which derivative at $r > t$ is $f'(r) = 1 - \frac{r}{\sqrt{r^2 - t}}$. Moreover,

$$f(t) \geq \frac{t}{2r}. \tag{3.13}$$

Taking $t := \frac{1}{4}\|\overline{a} - \overline{b}\|^2$, in the definition of f, the inequality (3.12) can be written as

$$f(d) \leq f(r) + c(\Delta)\varepsilon,$$

or

$$f(d) - f(r) \leq c(\Delta)\varepsilon.$$

The convexity and differentiability of f yield that

$$f'(r)(d - r) \leq c(\Delta)\varepsilon.$$

The latter reads

$$(r - d)\left(r - \sqrt{r^2 - \frac{\|\overline{a} - \overline{b}\|^2}{4}}\right) \leq c(\Delta)\varepsilon\sqrt{r^2 - \frac{\|\overline{a} - \overline{b}\|^2}{4}}. \tag{3.14}$$

Using that $\left(r - \sqrt{r^2 - \frac{\|\overline{a} - \overline{b}\|^2}{4}}\right) \geq \frac{\|\overline{a} - \overline{b}\|^2}{8r}$, see (3.13), and that $\sqrt{r^2 - \frac{\|\overline{a} - \overline{b}\|^2}{4}} \leq r$, from (3.14) we obtain

$$(r - d)\frac{\|\overline{a} - \overline{b}\|^2}{8r} \leq rc(\Delta)\varepsilon.$$

As $r-d>\Delta$, see (3.6),

$$\|\overline{a}-\overline{b}\|^2 \leq \frac{8r^2}{\Delta}c(\Delta)\varepsilon,$$

hence

$$\|\overline{a}-\overline{b}\| \leq 2r\sqrt{\frac{2c(\Delta)}{\Delta}}\sqrt{\varepsilon}.$$

From the latter and $\varepsilon<\Delta$ we get

$$\|a-b\| \leq \|\overline{a}-\overline{b}\|+2\varepsilon \leq \left(2r\sqrt{\frac{2c(\Delta)}{\Delta}}+2\sqrt{\varepsilon}\right)\sqrt{\varepsilon} \leq \left(2r\sqrt{\frac{2c(\Delta)}{\Delta}}+2\sqrt{\Delta}\right)\sqrt{\varepsilon}.$$

Setting $k=k(\Delta):=2r\sqrt{\frac{2c(\Delta)}{\Delta}}+2\sqrt{\Delta}$, we obtain for $a,b\in\varepsilon-\operatorname{argmin} d_C(x)$ with $\|a-b\|>\varepsilon$ that

$$\|a-b\| \leq k\sqrt{\varepsilon}.$$

As $k>\sqrt{\varepsilon}$, for $a,b\in C$, such that $a,b\in\varepsilon-\operatorname{argmin} d_C(x)$ with $\|a-b\|\leq\varepsilon$, obviously $\|a-b\|\leq k\sqrt{\varepsilon}$.

Therefore,

$$\operatorname{diam}\left(\varepsilon-\operatorname{argmin} d_C(x)\right) \leq k\sqrt{\varepsilon}. \tag{3.15}$$

This means that the projection mapping is single-valued on $B(x_0,\Delta)$, i.e. for $x\in B(x_0,\Delta)$ there exists unique point $p_C(x)\in C$ such that $d_C(x)=\|x-p_C(x)\|$. As $x_0\in T_C(r)$ was arbitrary, the projection mapping P_C is single-valued on $T_C(r)$.

It is routine to establish the continuity of the metric projection mapping P_C at x_0. Take $x,y\in B(x_0,\Delta/4)$. For their projections we have that $\|x-p_C(x)\|=d_C(x)$, $\|y-p_C(y)\|=d_C(y)$. Since

$$\|p_C(y)-x\| \leq \|p_C(y)-y\|+\|y-x\| = d_C(y)+\|y-x\| \leq d_C(x)+2\|y-x\|,$$

we have that $p_C(y)\in(2\|y-x\|)-\operatorname{argmin} d_C(x)$. Obviously $p_C(x)\in(2\|y-x\|)-\operatorname{argmin} d_C(x)$. For $\varepsilon:=2\|y-x\|$ we have that $\varepsilon<\Delta$. So, we can apply (3.15) to get that

$$\|p_C(y)-p_C(x)\| \leq \sqrt{2}k\sqrt{\|y-x\|}. \tag{3.16}$$

The latter yields that P_C is norm-to-norm continuous at x_0, and as x_0 was arbitrary in $T_C(r)$, on $T_C(r)$. From Theorem 2.2(c) it holds that C is r-prox-regular, thus completing the proof of (c) $\Rightarrow$ (a). □

From the proof of Theorem 1.1 it is clear that the property: the projection mapping P_C is single-valued and norm-to-norm continuous on $T_C(r)$ also characterizes r-prox-regular closed set C, but it is an external characterization.

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