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On estimating the rate of best trigonometric approximation by a modulus of smoothness

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Abstract

Best trigonometric approximation in L_p, $1 \leq p \leq \infty$, is characterized by a modulus of smoothness, which is equivalent to zero if the function is a trigonometric polynomial of a given degree. The characterization is just similar to the one given by the classical modulus of smoothness. The modulus possesses properties similar to those of the classical one.

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1 Introduction

Let $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, be the space of the 2π-periodic functions with finite L_p-norm on the circle $\mathbb{T}$ and T_n denote the set of the trigonometric polynomials of degree at most n. The best trigonometric approximation of a function $f \in L_p(\mathbb{T})$ is given by

$$E_n^T(f)_p = \inf_{\tau \in T_n} \|f - \tau\|_p,$$

where we have denoted by $\|\cdot\|_p$ the L_p-norm on $\mathbb{T}$.

The rate of best trigonometric approximation of $f \in L_p(\mathbb{T})$ can be nicely estimated by the classical moduli of smoothness of order $r \in \mathbb{N}$, defined by

$$\omega_r(f,t)_p = \sup_{0<h\leq t} \|\Delta_h^r f\|_p, \tag{1.1}$$

where the centred finite difference of order $r \in \mathbb{N}$ of f is given by

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h).$$

D. Jackson, S. N. Bernstein, A. Zygmund and S. B. Stechkin showed that (see for example [5, Ch. 7])

$$\begin{aligned} &E_n^T(f)_p \leq c\,\omega_r(f, n^{-1})_p, \\ &\omega_r(f,t)_p \leq c\,t^r \sum_{0\leq k\leq 1/t} (k+1)^{r-1} E_k^T(f)_p, \quad 0 < t \leq t_0. \end{aligned} \tag{1.2}$$

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Above and in what follows we denote by c positive constants, which do not depend on the functions in the relations, nor on $n \in \mathbb{N}$ or $0 < t \le t_0$; they may differ at each occurrence.

Thus the behaviour of the modulus of smoothness reveals to a great extent how fast the sequence of the trigonometric polynomials of best L_p-approximation converges to the function. However, there is one discrepancy – $E_n^T(f)_p$ is zero always when f is a trigonometric polynomial of degree n, whereas $\omega_r(f,t)_p$ is zero only if f is a constant, or to put it otherwise, $E_n^T(f)_p$ does not change its value when a trigonometric polynomial of degree n is added to the approximated function, whereas $\omega_r(f,t)_p$ does except when this polynomial is of degree 0. Naturally arises the problem of defining another modulus of smoothness, which describes the rate of best approximation by trigonometric polynomials in L_p like the classical one in (1.2) but in addition is equivalent to zero when the function is a trigonometric polynomial of a given degree. In [6] one solution to this problem was given. In this paper we shall discuss another definition of such a modulus.

Shevaldin defined in [13] (see also [12]) a finite difference operator whose kernel coincides with that of a linear differential operator with constant coefficients. In particular, the differential operator whose kernel is the set of trigonometric polynomials of degree $r-1$ is

$$\widetilde{D}_r = D_{r-1} \cdots D_1 \frac{d}{dx}, \quad D_j = \frac{d^2}{dx^2} + j^2 I,$$

where I is the identity. We can define a finite difference for $f \in L_p(\mathbb{T})$ which is identically zero only if $f \in T_{r-1}$ (see [13]) by

$$\widetilde{\Delta}_{r,h} f(x) = \Delta_{r-1,h} \cdots \Delta_{1,h} \Delta_{0,h} f(x), \tag{1.3}$$

where

$$\Delta_{j,h} f(x) = f(x+h) - 2\cos jh \cdot f(x) + f(x-h), \quad j = 1, 2, \ldots,$$

and $\Delta_{0,h} f(x) = \Delta_h f(x) = f(x+h/2) - f(x-h/2)$ is the classical centred finite difference of first order. (Note that a more general finite difference operator is defined in Shevaldin [14].) Now, let us set

$$\tilde{\omega}_r^T(f,t)_p = \sup_{0<h\le t} \|\widetilde{\Delta}_{r,h} f\|_p.$$

Note that $\tilde{\omega}_1^T(f,t)_p$ coincides with the classical modulus of continuity defined in (1.1) with $r = 1$.

We have

$$\tilde{\omega}_r^T(f,t)_p \equiv 0 \quad \Longleftrightarrow \quad f \in T_{r-1}.$$

The latter follows from the equivalence in Theorem 4.2 below and the fact that $\widetilde{D}_r f = 0$ if and only if $f \in T_{r-1}$.

We shall establish the following characterization of $E_n^T(f)_p$ by the trigonometric modulus of smoothness $\tilde{\omega}_r^T(f,t)_p$.

Theorem 1.1. *Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ and $r \in \mathbb{N}$. Then*

$$E_n^T(f)_p \le c\,\tilde{\omega}_r^T(f, n^{-1})_p, \quad n \ge r-1,$$

and

$$\tilde{\omega}_r^T(f,t)_p \le c\,t^{2r-1} \sum_{r-1 \le k \le 1/t} (k+1)^{2r-2} E_k^T(f)_p, \quad 0 < t \le \frac{1}{r}.$$

Relations (1.2) and Theorem 1.1 show that both $\omega_{2r-1}(f,t)_p$ and $\tilde{\omega}_r^T(f,t)_p$ give the same big $\mathcal{O}$ rate for the best trigonometric approximation, but the $\mathcal{O}$-constant in the estimate with $\tilde{\omega}_r^T(f,t)_p$ (or the modulus defined in [6]) can be substantially smaller for a particular function (see Remark 4.5). However, this is not true in general – the smallest constant c in the first inequality of Theorem 1.1 in $L_2(\mathbb{T})$ is at least as large, roughly speaking, as the one in the classical estimate with $\omega_{2r-1}(f,t)_p$ (see Remark 4.6).

Let us note that the Jackson-type estimate of Theorem 1.1 was established for the Hilbert space $L_2(\mathbb{T})$ by Babenko, Chernykh and Shevaldin [2] as estimates for the best constant on the right side were also given, and for $p=\infty$, $r=2$ by Shevaldin [15]. Our proof is based on a different approach and treats the general case.

The contents of the paper are organized as follows. In Section 2 we discuss properties of the finite differences $\widetilde{\Delta}_{r,h}$. In Section 3 we establish that $\tilde{\omega}_r^T(f,t)_p$ has very similar properties like the classical modulus of smoothness. Finally, in Section 4 we give a proof of Theorem 1.1.

2 The explicit form of $\widetilde{\Delta}_{r,h} f(x)$

The definition of the finite difference $\widetilde{\Delta}_{r,h}$ in (1.3) implies that there exist real numbers $c_{r,\ell}(h)$, $\ell = 0,1,\dots,2r-1$, which depend on the step h (continuously) such that

$$\widetilde{\Delta}_{r,h} f(x) = \sum_{\ell=0}^{2r-1} (-1)^\ell c_{r,\ell}(h)\, f\left(x + \frac{2(r-\ell)-1}{2}\,h\right).$$

We set for technical convenience $c_{r,\ell}(h) \equiv 0$ for $\ell < 0$ or $\ell > 2r-1$.

Lemma 2.1. *The coefficients $c_{r,\ell}(h)$ satisfy the recursion relation:*

(a) $c_{r+1,\ell}(h) = c_{r,\ell}(h) + 2\cos rh \cdot c_{r,\ell-1}(h) + c_{r,\ell-2}(h), \quad \ell = 0,1,\dots,2r+1,$

(b) $c_{r,0}(h) = c_{r,2r-1}(h) \equiv 1.$

Proof. The assertion follows by induction on r directly from

$$\widetilde{\Delta}_{r+1,h} f(x) = \Delta_{r,h}\big(\widetilde{\Delta}_{r,h} f\big)(x)$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{2r-1} (-1)^\ell c_{r,\ell}(h)\, f\left(x + \frac{2(r+1-\ell)-1}{2}\, h\right) \\
&\quad + 2\cos rh \sum_{\ell=1}^{2r} (-1)^\ell c_{r,\ell}(h)\, f\left(x + \frac{2(r+1-\ell)-1}{2}\, h\right) \\
&\quad + \sum_{\ell=2}^{2r+1} (-1)^\ell c_{r,\ell}(h)\, f\left(x + \frac{2(r+1-\ell)-1}{2}\, h\right). \qquad \square
\end{aligned}
$$

$\square$

Using the lemma above we prove by induction the following properties of $c_{r,\ell}(h)$.

Proposition 2.2. *The coefficients $c_{r,\ell}(h)$, $\ell = 0, 1, \dots, 2r-1$, $r \in \mathbb{N}$, $h \in \mathbb{R}$, satisfy the assertions:*

(i) *As a function of h, $c_{r,\ell}(h)$ is an even trigonometric polynomial of exact degree $\ell(2r-1-\ell)/2$;*

(ii) $c_{r,\ell}(h) = c_{r,2r-1-\ell}(h)$;

(iii) $|c_{r,\ell}(h)| \le \binom{2r-1}{\ell}$;

(iv) $c_{r,\ell}(0) = \binom{2r-1}{\ell}$.

Proof. Assertion (i) is trivial for $r = 1$. Assume that it is true for some $r \in \mathbb{N}$. Then Lemma 2.1 implies that $c_{r+1,\ell}(h)$ is an even trigonometric polynomial for each $\ell = 0, 1, \dots, 2r+1$. Further, by (b) of Lemma 2.1 we have $c_{r+1,0}(h) = c_{r+1,2r+1}(h) \equiv 1$. Next, for $\ell = 1, \dots, 2r$ the induction hypothesis gives that the degrees of $c_{r,\ell-2}(h)$ and $c_{r,\ell}(h)$ are less than $\ell(2r+1-\ell)/2$, whereas the exact degree of $c_{r,\ell-1}(h)$ is $(\ell-1)(2r-\ell)/2$. Now, relation (a) of Lemma 2.1 implies that $c_{r+1,\ell}(h)$ is of exact degree $r + (\ell-1)(2r-\ell)/2 = \ell(2r+1-\ell)/2$.

To establish (ii) we first observe that since $\Delta_{j,-h} f(x) = \Delta_{j,h} f(x)$ for $j \in \mathbb{N}_0$, then $\widetilde{\Delta}_{r,-h} f(x) = \widetilde{\Delta}_{r,h} f(x)$. Also, as we have already noted, $c_{r,\ell}(-h) = c_{r,\ell}(h)$. Hence we infer that for any continuous function f and real h there holds

$$
\begin{aligned}
&\sum_{\ell=0}^{2r-1} (-1)^\ell c_{r,\ell}(h)\, f\left(\frac{2(r-\ell)-1}{2}\, h\right) \\
&\qquad = \widetilde{\Delta}_{r,h} f(0) = \widetilde{\Delta}_{r,-h} f(0) \\
&\qquad = \sum_{\ell=0}^{2r-1} (-1)^\ell c_{r,\ell}(-h)\, f\left(-\frac{2(r-\ell)-1}{2}\, h\right) \\
&\qquad = \sum_{\ell=0}^{2r-1} (-1)^\ell c_{r,\ell}(h)\, f\left(\frac{2(r-(2r-1-\ell))-1}{2}\, h\right)
\end{aligned}
$$

$$= \sum_{\ell=0}^{2r-1} (-1)^\ell c_{r,2r-1-\ell}(h)\, f\left(\frac{2(r-\ell)-1}{2}\, h\right),$$

as at the last step we have substituted ℓ with $2r-1-\ell$. Consequently, for every continuous function f and real h we have

$$\sum_{\ell=0}^{2r-1} (-1)^\ell [c_{r,\ell}(h) - c_{r,2r-1-\ell}(h)]\, f\left(\frac{2(r-\ell)-1}{2}\, h\right) = 0.$$

Hence (ii) follows.

Assertions (iii) and (iv) follow by induction on r as we take into consideration Lemma 2.1, relation (ii) and the trivial identities

$$\binom{2r-1}{1} + 2 = \binom{2r+1}{1}$$

and

$$\binom{2r-1}{\ell} + 2\binom{2r-1}{\ell-1} + \binom{2r-1}{\ell-2} = \binom{2r+1}{\ell}$$

for $\ell = 2, \ldots, r$. $\square$

Let us set

$$P_k(h) = \begin{cases} \prod_{j=1}^{k} \sin \frac{jh}{2}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The next assertion contains the explicit form of the coefficients $c_{r,\ell}(h)$.

Proposition 2.3. *For $\ell = 0, 1, \ldots, 2r-1$, $r \in \mathbb{N}$ and $h \in \mathbb{R}$ we have*

$$c_{r,\ell}(h) = \frac{P_{2r-1}(h)}{P_\ell(h) P_{2r-1-\ell}(h)}$$

as for $h = 0$ the right side is defined by continuity.

Proof. We use induction on r. Obviously for every $r \in \mathbb{N}$ and $\ell = 0$ or $\ell = 2r-1$ we have $c_{r+1,0}(h) = c_{r,0}(h) = c_{r+1,2r+1}(h) = c_{r,2r-1}(h) = 1$.

For $\ell = 1$ we have by Lemma 2.1, (a)-(b),

$$c_{r+1,1}(h) = c_{r,1}(h) + 2\cos rh = \frac{\sin(2r-1)\frac{h}{2}}{\sin\frac{h}{2}} + 2\cos rh = \frac{\sin(2r+1)\frac{h}{2}}{\sin\frac{h}{2}}.$$

Let now $\ell = 2, \ldots, 2r-1$. Then, using relation (a) of Lemma 2.1, we get

$$c_{r+1,\ell}(h) = c_{r,\ell}(h) + 2c_{r,\ell-1}(h)\cos rh + c_{r,\ell-2}(h)$$

$$
\begin{aligned}
&= \frac{P_{2r-1}(h)}{P_{\ell-1}(h)P_{2r-\ell}(h)} \left(\frac{\sin(2r-\ell)\frac{h}{2}}{\sin \ell \frac{h}{2}} + 2\cos rh + \frac{\sin(\ell-1)\frac{h}{2}}{\sin(2r+1-\ell)\frac{h}{2}} \right) \\
&= \frac{P_{2r-1}(h)}{P_{\ell-1}(h)P_{2r-\ell}(h)} \; \frac{\sin(2r-\ell)\frac{h}{2} + \sin \ell \frac{h}{2} \cos rh}{\sin \ell \frac{h}{2}} \\
&\quad + \frac{P_{2r-1}(h)}{P_{\ell-1}(h)P_{2r-\ell}(h)} \; \frac{\sin(\ell-1)\frac{h}{2} + \sin(2r+1-\ell)\frac{h}{2}\cos rh}{\sin(2r+1-\ell)\frac{h}{2}} \\
&= \frac{P_{2r}(h)}{P_{\ell-1}(h)P_{2r-\ell}(h)} \left(\frac{\cos \ell \frac{h}{2}}{\sin \ell \frac{h}{2}} + \frac{\cos(2r+1-\ell)\frac{h}{2}}{\sin(2r+1-\ell)\frac{h}{2}} \right) \\
&= \frac{P_{2r}(h)}{P_{\ell-1}(h)P_{2r-\ell}(h)} \; \frac{\sin(2r+1)\frac{h}{2}}{\sin \ell \frac{h}{2} \sin(2r+1-\ell)\frac{h}{2}} \\
&= \frac{P_{2r+1}(h)}{P_{\ell}(h)P_{2r+1-\ell}(h)}.
\end{aligned}
$$

The case $\ell = 2r$ is symmetric to $\ell = 1$ and the statement follows from the equality $c_{r+1,2r}(h) = c_{r+1,1}(h)$ (see assertion (ii) of Proposition 2.2). $\square$

Remark 2.4. Let us mention that the formula of Proposition 2.3 can also be verified by means of the relations given in [11, Remark 10.2].

The properties above and especially the last one show that the coefficients $c_{r,\ell}(h)$ are very similar to the classical binomial coefficients but unlike them depend on one more parameter – h.

Now we turn to integral representations of $\Delta_{j,h}$ and $\widetilde{\Delta}_{r,h}$. Let $f * g$ denote the convolution of the functions $f, g \in L_1(\mathbb{T})$, defined by

$$
f * g(x) = \int_{\mathbb{T}} f(x-y)\, g(y)\, dy, \quad x \in \mathbb{T},
$$

and $\hat{f}(k)$, $k \in \mathbb{Z}$, denote the Fourier coefficients of $f \in L_1(\mathbb{T})$, defined by

$$
\hat{f}(k) = \int_{\mathbb{T}} f(x)\, e^{-ikx}\, dx, \quad k \in \mathbb{Z}.
$$

We omit the constant multipliers that are usually included in the definitions of the convolution and the Fourier transform for convenience in the subsequent considerations.

For $0 < h < 2\pi$ we define the 2π-periodic function $B_{0,h}$ by setting for $x \in [-\pi, \pi]$

$$
B_{0,h}(x) = \begin{cases} \dfrac{1}{h}, & x \in [-h/2, h/2], \\ 0, & x \in [-\pi,\pi] \backslash [-h/2, h/2]; \end{cases}
$$

and for $j \in \mathbb{N}$ and $0 < h < \pi$ we define the 2π-periodic function $B_{j,h}$ by setting for $x \in [-\pi, \pi]$

$$
B_{j,h}(x) = \frac{1}{jh^2} \sin[j(h - |x|)_+].
$$

Next, for $r \in \mathbb{N}$ and $0 < h < 2\pi/(2r-1)$ we define the 2π-periodic function $B^T_{j,h}$ by setting

$$B^T_{r,h}(x) = B_{0,h} * B_{1,h} * \cdots * B_{r-1,h}(x).$$

The functions $B^T_{r,h}$ are trigonometric B-splines of order $2r-1$ and nodes at $jh/2$, $j = 1-2r, \ldots, 2r-1$. The trigonometric B-splines have been introduced by Schoenberg [10] (see also [11, § 10.8]).

Let $W^s_p(\mathbb{T})$, $s \in \mathbb{N}$, denote the Sobolev spaces of 2π-periodic functions, that is,

$$W^s_p(\mathbb{T}) = \{g \in L_p(\mathbb{T}) \ : \ g, g', \ldots, g^{(s-1)} \in AC(\mathbb{T}), \ g^{(s)} \in L_p(\mathbb{T})\},$$

where $AC(\mathbb{T})$ is the set of the 2π-periodic absolutely continuous functions. The following representations of $\Delta_{j,h}$ and $\widetilde{\Delta}_{r,h}$ hold true.

Proposition 2.5. *Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ and $j \in \mathbb{N}$. Then we have*

$$\Delta_{j,h} f(x) = h^2 D_j(B_{j,h} * f)(x) \tag{2.1}$$

and hence if $f \in W^2_p(\mathbb{T})$, then

$$\Delta_{j,h} f(x) = h^2 B_{j,h} * D_j f(x). \tag{2.2}$$

Proof. It is sufficient to verify (2.1). We just have

$$\begin{aligned}
h^2 B_{j,h} * f(x) &= \frac{1}{j}\int_{-h}^{h} \sin j(h-|y|) f(x-y)\, dy \\
&= \frac{1}{j}\int_{-h}^{0} \sin j(h+y) f(x-y)\, dy + \frac{1}{j}\int_{0}^{h} \sin j(h-y) f(x-y)\, dy \\
&= \frac{1}{j}\int_{x}^{x+h} \sin j(x+h-u) f(u)\, du + \frac{1}{j}\int_{x-h}^{x} \sin j(h-x+u) f(u)\, du.
\end{aligned}$$

Next, we consecutively calculate

$$\begin{aligned}
&h^2 \frac{d}{dx} B_{j,h} * f(x) \\
&\qquad = \int_{x}^{x+h} \cos j(x+h-u) f(u)\, du - \int_{x-h}^{x} \cos j(h-x+u) f(u)\, du
\end{aligned}$$

and

$$\begin{aligned}
&h^2 \left(\frac{d}{dx}\right)^2 B_{j,h} * f(x) \\
&\qquad = f(x+h) - \cos jh \cdot f(x) - j\int_{x}^{x+h} \sin j(x+h-u) f(u)\, du \\
&\qquad\quad - \cos jh \cdot f(x) + f(x-h) - j\int_{x-h}^{x} \sin j(h-x+u) f(u)\, du \\
&\qquad = \Delta_{j,h} f(x) - j^2 h^2 B_{j,h} * f(x).
\end{aligned}$$

Hence relation (2.1) follows. □

Iterating (2.1) and taking into account the trivial fact that

$$\Delta_h f(x) = h\,\frac{d}{dx}(B_{0,h} * f)(x),$$

we get the following assertion.

Proposition 2.6. *Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ and $r \in \mathbb{N}$. Then we have*

$$\widetilde{\Delta}_{r,h} f(x) = h^{2r-1}\widetilde{D}_r(B^T_{r,h} * f)(x)$$

and hence if $f \in W_p^{2r-1}(\mathbb{T})$, then

$$\widetilde{\Delta}_{r,h} f(x) = h^{2r-1} B^T_{r,h} * \widetilde{D}_r f(x).$$

Finally, let us also point out the representation of $\widetilde{\Delta}_{r,h}$ by a multiple integral:

$$\widetilde{\Delta}_{r,h} f(x) = \frac{1}{(r-1)!}\,\widetilde{D}_r \int_{-h/2}^{h/2}\int_{-h}^{h}\cdots\int_{-h}^{h}\prod_{j=1}^{r-1}\sin j(h-|y_j|) \\ \times f(x-(y_0+\cdots+y_{r-1}))\,dy_0\,dy_1\ldots\,dy_{r-1}.$$

3 Properties of $\tilde{\omega}_r^T(f,t)_p$

The modulus $\tilde{\omega}_r^T(f,t)_p$ retains the properties of the classical one. They are the following:

1. $\tilde{\omega}_r^T(f+g,t)_p \le \tilde{\omega}_r^T(f,t)_p + \tilde{\omega}_r^T(g,t)_p$ for $f,g \in L_p(\mathbb{T})$;
2. $\tilde{\omega}_r^T(cf,t)_p = |c|\,\tilde{\omega}_r^T(f,t)_p$, c is a constant;
3. $\tilde{\omega}_r^T(f,t)_p \le \tilde{\omega}_r^T(f,t')_p,\ t \le t'$;
4. $\tilde{\omega}_r^T(f,t)_p \to 0$ as $t \to 0$;
5. $\tilde{\omega}_r^T(f,t)_p \le 4\,\tilde{\omega}_{r-1}^T(f,t)_p,\ r \ge 2$;
6. $\tilde{\omega}_1^T(f,t)_p \le 2\|f\|_p,\ f \in L_p(\mathbb{T})$, and $\tilde{\omega}_1^T(f,t)_p \le t\,\|f'\|_p,\ f \in W_p^1(\mathbb{T})$ ($\tilde{\omega}_1^T(f,t)_p$ coincides with the ordinary modulus of continuity);
7. $\tilde{\omega}_r^T(f,\lambda t)_p \le (\lambda+1)^{2r-1}\tilde{\omega}_r^T(f,t)_p$, $\lambda > 0$;
8. $\tilde{\omega}_r^T(f,t)_p \le t^2\,\tilde{\omega}_{r-1}^T(D_{r-1}f,t)_p,\ f \in W_p^2(\mathbb{T}),\ r \ge 2$;
9. The Marchaud inequality

$$\tilde{\omega}_r^T(f,t)_p \le c\,t^{2r-1}\left(\int_t^{t_0}\frac{\tilde{\omega}_{r+1}^T(f,u)_p}{u^{2r}}\,du + \|f\|_p\right), \quad 0 < t \le t_0.$$

Only the proof of relations 7, 8 and 9 needs somewhat more considerations.

Proof of Property 7. Set for $j \in \mathbb{Z}$ and $h \in \mathbb{R}$

$$\widehat{\Delta}_{j,h} f(x) = f\left(x + \frac{h}{2}\right) - e^{ijh} f\left(x - \frac{h}{2}\right).$$

Let $m \in \mathbb{N}$, as $m \geq 2$. In order to get a simple representation of $\Delta_{j,mh}$ by $\Delta_{j,h}$, we shall avail ourselves of the following expression of $\Delta_{j,h}$ in terms of the finite differences of first order defined above (cf. [12, 13]):

$$\Delta_{j,h} f(x) = \widehat{\Delta}_{j,h} \widehat{\Delta}_{-j,h} f(x). \tag{3.1}$$

Note also that

$$\widehat{\Delta}_{0,h} f(x) = \Delta_{0,h} f(x). \tag{3.2}$$

Direct calculations verify the relation

$$\widehat{\Delta}_{j,h_1+h_2} f(x) = \widehat{\Delta}_{j,h_2} f\left(x + \frac{h_1}{2}\right) + e^{ijh_2} \widehat{\Delta}_{j,h_1} f\left(x - \frac{h_2}{2}\right).$$

Setting $h_1 = h$ and $h_2 = (m-1)h$, we get

$$\widehat{\Delta}_{j,mh} f(x) = \widehat{\Delta}_{j,(m-1)h} f\left(x + \frac{h}{2}\right) + e^{ij(m-1)h} \widehat{\Delta}_{j,h} f\left(x - \frac{(m-1)h}{2}\right).$$

Iterating the latter, we arrive at

$$\widehat{\Delta}_{j,mh} f(x) = \sum_{\ell=0}^{m-1} e^{ij\ell h} \widehat{\Delta}_{j,h} f\left(x + \frac{m-2\ell-1}{2}\, h\right). \tag{3.3}$$

Now, by means of (1.3) and (3.1)-(3.3), we derive the representation

$$\begin{aligned}\widetilde{\Delta}_{r,mh} f(x) = \sum_{\ell_0=0}^{m-1} \sum_{\ell_1=0}^{m-1} \cdots \sum_{\ell_{2r-2}=0}^{m-1} \exp\left(ih \sum_{j=1}^{r-1} j(\ell_{2j-1} - \ell_{2j})\right) \\ \times \widetilde{\Delta}_{r,h} f\left(x + h\left(\left(r - \frac{1}{2}\right)(m-1) - \sum_{j=0}^{2r-2} \ell_j\right)\right).\end{aligned}$$

Consequently,

$$\|\widetilde{\Delta}_{r,mh} f\|_p \leq \sum_{\ell_0=0}^{m-1} \sum_{\ell_1=0}^{m-1} \cdots \sum_{\ell_{2r-2}=0}^{m-1} \|\widetilde{\Delta}_{r,h} f\|_p;$$

hence

$$\tilde{\omega}_r^T(f, mt)_p \leq m^{2r-1} \tilde{\omega}_r^T(f, t)_p. \tag{3.4}$$

Finally, the property under consideration follows directly from Property 3 and (3.4) with $m = [\lambda] + 1$, where $[\lambda]$ denotes the largest integer not greater than λ. □

Proof of Property 8. By (1.3) and (2.2) we have

$$\widetilde{\Delta}_{r,h}f(x) = \Delta_{r-1,h}(\widetilde{\Delta}_{r-1,h}f)(x) = h^2 B_{r-1,h} * D_{r-1}(\widetilde{\Delta}_{r-1,h}f)(x) \\ = h^2 B_{r-1,h} * \widetilde{\Delta}_{r-1,h}(D_{r-1}f)(x). \tag{3.5}$$

Also, we have for $j \in \mathbb{N}$

$$\|B_{j,h}\|_1 = \frac{1}{jh^2}\int_{-h}^{h} |\sin j(h-|x|)|\, dx = \frac{2}{jh^2}\int_0^h |\sin j(h-x)|\, dx \\ \le \frac{2}{jh^2}\int_0^h j(h-x)\, dx = 1. \tag{3.6}$$

Now, (3.5), (3.6) and Young's inequality imply the property. $\square$

Property 9 follows from Theorem 1.1 by a standard argument (see e.g. [5, p. 210]).

Let us also mention the following properties of the modulus $\tilde{\omega}_r^T(f,t)_p$, which can be verified by means of Theorem 4.2 below and [6, Theorems 1.2 and 4.14].

Theorem 3.1. *Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ and $r \in \mathbb{N}$. We have*

(i) $\tilde{\omega}_r^T(f,t)_p = o(t^{2r-1})$ *if and only if* $f \in T_{r-1}$;

(ii) *If* $1 < p \le \infty$, *then* $\tilde{\omega}_r^T(f,t)_p = \mathcal{O}(t^{2r-1})$ *if and only if* $f \in W_p^{2r-1}(\mathbb{T})$;

(iii) $\tilde{\omega}_r^T(f,t)_1 = \mathcal{O}(t^{2r-1})$ *if and only if* $f \in W_1^{2r-3}(\mathbb{T})$ *and* $f^{(2r-2)}$ *is equivalent to a function of bounded variation.*

4 Proof of the characterization of $E_n^T(f)_p$ by $\tilde{\omega}_r(f,t)_p$

For $f \in L_p(\mathbb{T})$ and $t > 0$ we define the K-functional

$$K_r^T(f,t)_p = \inf_{g \in W_p^{2r-1}(\mathbb{T})} \big\{\|f-g\|_p + t^{2r-1}\|\widetilde{D}_r g\|_p\big\}. \tag{4.1}$$

The following characterization of $E_n^T(f)_p$ in terms of $K_r^T(f,t)_p$ was established in [6].

Theorem 4.1. *Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ and $r \in \mathbb{N}$. Then*

$$E_n^T(f)_p \le c\, K_r^T(f,n^{-1})_p, \quad n \ge r-1,$$

and

$$K_r^T(f,t)_p \le c\, t^{2r-1} \sum_{r-1 \le k \le 1/t} (k+1)^{2r-2} E_k^T(f)_p, \quad 0 < t \le \frac{1}{r}.$$

Thus to verify Theorem 1.1, it is sufficient to prove that the K-functional (4.1) and the modulus $\tilde{\omega}_r^T(f,t)_p$ are equivalent, that is, their ratio is bounded between two positive constants, which are independent of f and t. We shall denote that by $K_r^T(f,t)_p \sim \tilde{\omega}_r^T(f,t)_p$.

Theorem 4.2. *For $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$, $r \in \mathbb{N}$ and $0 < t \le t_0$ we have*

$$K_r^T(f,t)_p \sim \tilde{\omega}_r^T(f,t)_p.$$

For the proof we need the following auxiliary result.

Lemma 4.3. *Let $r \in \mathbb{N}$ and $q_1, q_2, \dots, q_{2r-1}$ be different prime numbers. Set $q_0 = 1$. For $0 \le t \le \pi/(2r)$ and $x \ge 2$ we have*

$$x^{4r-2} \sum_{m=0}^{2r-1} \frac{1}{q_m} \prod_{j=1-r}^{r-1} \sin^2 \frac{\sqrt{q_m}(x+tj)}{2} \ge c > 0.$$

Proof. Suppose that the assertion is not valid. Then, since the expression on the left hand-side above is a positive continuous function of (x,t), there exist sequences $\{x_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ and integers $j_m \in [1-r, r-1]$, $m = 0, 1, \dots, 2r-1$, such that

$$\lim_{n\to\infty} x_n = \infty, \tag{4.2}$$

$$0 \le t_n \le \pi/(4r), \quad n \in \mathbb{N}, \tag{4.3}$$

and

$$\lim_{n\to\infty} x_n \sin \sqrt{q_m}(x_n + j_m t_n) = 0, \quad m = 0, 1, \dots, 2r-1. \tag{4.4}$$

Since there are $2r-1$ integers in the interval $[1-r, r-1]$ and the j_m's are $2r$ in number, then at least two of them are equal. Assume that $j_{m'} = j_{m''} = j$ and set $y_n = \sqrt{q_{m'}}(x_n + jt_n)$ and $q = q_{m''}/q_{m'}$. Then as we take into account (4.2), (4.3) and (4.4) with $m = m'$ and $m = m''$, we deduce that

$$\lim_{n\to\infty} y_n = \infty,$$
$$\lim_{n\to\infty} y_n \sin y_n = 0$$

and

$$\lim_{n\to\infty} y_n \sin y_n \sqrt{q} = 0.$$

These relations imply that there exist two sequences of positive integers $\{k_n\}_{n=1}^\infty$ and $\{\ell_n\}_{n=1}^\infty$ and two sequences of real numbers $\{\varepsilon_n\}_{n=1}^\infty$ and $\{\eta_n\}_{n=1}^\infty$ such that

$$y_n = k_n\pi + \varepsilon_n = \frac{\ell_n \pi}{\sqrt{q}} + \eta_n, \tag{4.5}$$

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \ell_n = \infty, \tag{4.6}$$

$$\lim_{n\to\infty} \frac{\ell_n}{k_n} = \sqrt{q} \tag{4.7}$$

and

$$\lim_{n\to\infty} k_n\varepsilon_n = \lim_{n\to\infty} k_n\eta_n = 0. \tag{4.8}$$

Then, since $\sqrt{q}$ is irrational, $qk_n^2 \neq \ell_n^2$ for all $n \in \mathbb{N}$ and by (4.5)-(4.8) we arrive at the contradiction:

$$1 \leq |qk_n^2 - \ell_n^2| = (k_n\sqrt{q} + \ell_n)|k_n\sqrt{q} - \ell_n| = k_n\, o(k_n^{-1}) = o(1).$$

Thus the validity of the lemma is verified. $\square$

Remark 4.4. For $r = 1$ it is sufficient to take only two summands in the formulation of the lemma. However, this is not valid for $r \geq 2$. Indeed, let $\sqrt{q}$ be an irrational. Then, as is known (see e.g. [8, Ch. 11]), there exist two sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ and $\{\ell_n\}_{n=1}^{\infty}$, tending to infinity, such that

$$0 < \sqrt{q} - \frac{\ell_n}{k_n} < \frac{1}{k_n^2}, \quad n \in \mathbb{N}.$$

Set

$$x_n = k_n\pi + \frac{1}{k_n^2} \to \infty \quad \text{as} \quad n \to \infty$$

and

$$t_n = \frac{\pi k_n(k_n\sqrt{q} - \ell_n)}{k_n\sqrt{q}} \to 0 \quad \text{as} \quad n \to \infty.$$

Then

$$\lim_{n\to\infty} x_n \sin(x_n + jt_n) = j\pi,$$
$$\lim_{n\to\infty} x_n \sin\sqrt{q}(x_n - t_n) = 0$$

and

$$|x_n \sin\sqrt{q}(x_n + jt_n)| \leq c, \quad n \in \mathbb{N}.$$

However, it seems that we can do with three summands in the case $r \geq 2$, but in our opinion this demands more complicated considerations, which is superfluous in the context of this paper. A similar argument shows that the power of x in the formulation of the lemma cannot be decreased. Also, it is clear that no one of the irrational multipliers in the argument of the sines can be replaced with a rational one.

We proceed to the proof of Theorem 4.2.

Proof of Theorem 4.2. Properties 1, 5, 6 and 8 imply for any $g \in W_p^{2r-1}(\mathbb{T})$

$$\begin{aligned}\tilde{\omega}_r^T(f,t)_p &\le \tilde{\omega}_r^T(f-g,t)_p + \tilde{\omega}_r^T(g,t)_p \\ &\le 2^{2r-1}\left(\|f-g\|_p + t^{2r-1}\|\widetilde{D}_r g\|_p\right).\end{aligned}$$

Hence, taking the infimum on $g \in W_p^{2r-1}(\mathbb{T})$ we get the inequality

$$\tilde{\omega}_r^T(f,t)_p \le 2^{2r-1} K_r^T(f,t)_p.$$

To establish the converse estimate, we shall construct for $f \in L_p(\mathbb{T})$ and $0 < t \le \pi/(2r)$ a function $g_t \in W_p^{2r-1}(\mathbb{T})$ such that

$$\|f-g_t\|_p \le c\,\tilde{\omega}_r^T(f,t)_p \tag{4.9}$$

and

$$t^{2r-1}\|\widetilde{D}_r g_t\|_p \le c\,\tilde{\omega}_r^T(f,t)_p, \tag{4.10}$$

where c is a constant whose value does not depend on f or $0 < t \le \pi/(2r)$. Inequalities (4.9)-(4.10) imply immediately

$$K_r^T(f,t)_p \le c\,\tilde{\omega}_r^T(f,t)_p, \quad 0 < t \le \pi/(2r). \tag{4.11}$$

For $t_0 > \pi/(2r)$ this relation is extended to $0 < t \le t_0$ by means of

$$\begin{aligned}K_r^T(f,t)_p &\le \frac{2rt_0}{\pi}\, K_r^T\left(f, \frac{\pi t}{2rt_0}\right)_p \le c\,\tilde{\omega}_r^T\left(f, \frac{\pi t}{2rt_0}\right)_p \\ &\le c\,\tilde{\omega}_r^T(f,t)_p,\end{aligned}$$

as at the second estimate we have applied (4.11) and at the last one Property 3 of the modulus.

So, let $0 < t \le \pi/(2r)$. We define the kernel $A_{r,t} \in L_1(\mathbb{T})$ in such a way that we have

$$A_{r,t} * f(x) = a_r \sum_{\ell=1}^{2r-1} (-1)^{\ell-1} \int_{-1}^{1} \big(|y|(1-|y|)\big)^{s_r} c_{r,\ell}(ty) f(x-\ell t y)\, dy \tag{4.12}$$

with $a_r = (2s_r+1)!/(2[s_r!]^2)$ and $s_r = 16r-5$. Note that $0 < t \le \pi/(2r)$ implies $(2r-1)t < \pi$ and hence such a 2π-periodic kernel $A_{r,t}$ exists. We set $g_t = A_{r,t} * f$. Then

$$f(x) - g_t(x) = a_r \int_{-1}^{1} \big(|y|(1-|y|)\big)^{s_r} \widetilde{\Delta}_{r,ty} f(x-(2r-1)y/2)\, dy,$$

and hence, applying the generalized Minkowski inequality, we conclude that (4.9) is satisfied with $c = 1$.

Further, we shall show that there exist functions $C_{r,t} \in L_1(\mathbb{T})$ for $0 < t \leq \pi/(2r)$, such that

$$A_{r,t} = C_{r,t} * \sum_{m=0}^{2r-1} B^T_{r,t\sqrt{q_m}} * B^T_{r,t\sqrt{q_m}}, \tag{4.13}$$

where $q_0 = 1$ and q_m, $m = 1, 2, \ldots, 2r-1$, are different prime numbers, and

$$\|C_{r,t}\|_1 \leq c, \quad 0 < t \leq \pi/(2r). \tag{4.14}$$

Then Proposition 2.6 implies

$$\begin{aligned} t^{2r-1}\widetilde{D}_r g_t(x) &= C_{r,t} * \sum_{m=0}^{2r-1} q_m^{1/2-r} B^T_{r,t\sqrt{q_m}} * (t\sqrt{q_m})^{2r-1}\widetilde{D}_r(B^T_{r,t\sqrt{q_m}} * f)(x) \\ &= C_{r,t} * \sum_{m=0}^{2r-1} q_m^{1/2-r} B^T_{r,t\sqrt{q_m}} * \widetilde{\Delta}_{r,t\sqrt{q_m}} f(x); \end{aligned}$$

hence, in view of (3.6) and (4.14), we get (4.10) by means of Young's inequality.

Thus, it remains to verify that there exist kernels $C_{r,t} \in L_1(\mathbb{T})$ with (4.13)-(4.14). To this end, we shall apply Fourier transform methods. The Fourier coefficients of $B_{j,t}$ are

$$\begin{aligned} \widehat{B}_{0,t}(k) &= \frac{\sin(\frac{t}{2}\,k)}{\frac{t}{2}\,k}, \\ \widehat{B}_{j,t}(k) &= \frac{\sin[\frac{t}{2}(k+j)]}{\frac{t}{2}(k+j)}\,\frac{\sin[\frac{t}{2}(k-j)]}{\frac{t}{2}(k-j)}, \quad j > 0. \end{aligned} \tag{4.15}$$

They are calculated either directly, or, more easily, by taking the Fourier transform of both sides of (2.1).

Relations (4.15) yield

$$\widehat{B}^T_{r,t}(k) = \prod_{j=0}^{r-1} \widehat{B}_{j,t}(k) = \prod_{j=1-r}^{r-1} \frac{\sin[\frac{t}{2}(k+j)]}{\frac{t}{2}(k+j)}. \tag{4.16}$$

On the other hand, by applying the Fourier transform on both sides of (4.12), we get

$$\widehat{A}_{r,t}(k)\,\hat{f}(k) = a_r \sum_{\ell=1}^{2r-1} (-1)^{\ell-1} \int_{-1}^{1} \big(|y|(1-|y|)\big)^{s_r} c_{r,\ell}(ty) e^{-ik\ell ty} \hat{f}(k)\, dy;$$

hence

$$\begin{aligned} \widehat{A}_{r,t}(k) &= a_r \sum_{\ell=1}^{2r-1} (-1)^{\ell-1} \int_{-1}^{1} \big(|y|(1-|y|)\big)^{s_r} c_{r,\ell}(ty) e^{-ik\ell ty}\, dy \\ &= 2a_r \sum_{\ell=1}^{2r-1} (-1)^{\ell-1} \int_{0}^{1} \big(y(1-y)\big)^{s_r} c_{r,\ell}(ty) \cos(k\ell ty)\, dy. \end{aligned} \tag{4.17}$$

Above we have also taken into consideration that $c_{r,\ell}(h)$ are even functions. We set for $0 < t \le \pi/(2r)$

$$v_t(k) = \frac{\widehat{A}_{r,t}(k)}{\sum\limits_{m=0}^{2r-1} \left(\widehat{B}^T_{r,t\sqrt{q_m}}(k)\right)^2}, \quad k \in \mathbb{Z}.$$

Now, in view of (4.16)-(4.17), in order to show that there exist kernels $C_{r,t} \in L_1(\mathbb{T})$ with (4.13)-(4.14), it remains to establish that $v_t(k)$, $k \in \mathbb{Z}$, are the Fourier coefficients of summable 2π-periodic functions with norms, which are uniformly bounded on $0 < t \le \pi/(2r)$. For this purpose, it is sufficient to show that the functions $v_t(k)$, $0 < t \le \pi/(2r)$, satisfy the following conditions (see e.g. [3, Corollary 6.3.9]):

(a) v_t are even functions on $\mathbb{Z}$ for each $0 < t \le \pi/(2r)$,

(b) $\lim_{k\to\infty} v_t(k) = 0$ for each $0 < t \le \pi/(2r)$,

(c) The quantities

$$\sum_{k=1}^{\infty} k|v_t(k+1) - 2v_t(k) + v_t(k-1)|$$

are uniformly bounded for $0 < t \le \pi/(2r)$.

Property (a) is clearly fulfilled. To establish the other two, we observe that

$$v_t(k) = u_t(tk), \quad k \ge 0$$

with

$$u_t(x) = \frac{2^{3-4r} a_r \prod\limits_{j=1-r}^{r-1} (x+tj)^2}{\sum\limits_{m=0}^{2r-1} \frac{1}{q_m} \prod\limits_{j=1-r}^{r-1} \sin^2 \frac{\sqrt{q_m}(x+tj)}{2}} \times \sum_{\ell=1}^{2r-1} (-1)^{\ell-1} \int_0^1 \big(y(1-y)\big)^{s_r} c_{r,\ell}(ty) \cos(\ell y x)\, dy.$$

Integration by parts gives for $x \ge 1$, $\ell = 1, \dots, 2r-1$ and $0 < t \le \pi/(2r)$

$$\begin{aligned} &\left| \int_0^1 \big(y(1-y)\big)^{s_r} c_{r,\ell}(ty) \cos(\ell y x)\, dy \right| \\ &= \frac{1}{(\ell x)^{s_r}} \left| \int_0^1 \Big(\big(y(1-y)\big)^{s_r} c_{r,\ell}(ty) \Big)^{(s_r)} \sin(\ell y x)\, dy \right| \\ &\le \frac{c}{x^{s_r}}. \end{aligned} \tag{4.18}$$

Similarly, we have for $x \geq 1$, $\ell = 1, \ldots, 2r-1$ and $0 < t \leq \pi/(2r)$

$$\left| \int_0^1 y\big(y(1-y)\big)^{s_r} c_{r,\ell}(ty) \sin(\ell yx)\, dy \right| \leq \frac{c}{x^{s_r}} \tag{4.19}$$

and

$$\left| \int_0^1 y^2\big(y(1-y)\big)^{s_r} c_{r,\ell}(ty) \cos(\ell yx)\, dy \right| \leq \frac{c}{x^{s_r}}. \tag{4.20}$$

Now, (4.18) and Lemma 4.3 imply (b).

Finally, to verify (c), we observe that $u_t \in W^2_\infty(\mathbb{R}_+)$ as, moreover, by the estimate

$$\frac{d^l}{dx^l} \left(\frac{\displaystyle\prod_{j=1-r}^{r-1} (x+tj)^2}{\displaystyle\sum_{m=0}^{2r-1} \frac{1}{q_m} \prod_{j=1-r}^{r-1} \sin^2 \frac{\sqrt{q_m}(x+tj)}{2}} \right) \leq c, \quad 0 \leq x \leq 2,$$

for $0 < t \leq \pi/(2r)$ and $l = 0, 1, 2$ together with (4.18)-(4.20) and Lemma 4.3, we get for all $0 < t \leq \pi/(2r)$ that

$$\|u_t''\|_{\infty[0,3]} \leq c$$

and

$$\|u_t''\|_{\infty[t(k-1),t(k+1)]} \leq \frac{c}{(tk)^3}, \quad k > [1/t].$$

Consequently, for all $0 < t \leq \pi/(2r)$ we have

$$\begin{aligned}
\sum_{k=1}^{\infty} k|v_t(k+1) - 2v_t(k) + v_t(k-1)| &\leq \sum_{k=1}^{\infty} k\, t^2 \|u_t''\|_{\infty[t(k-1),t(k+1)]} \\
&\leq t^2\, \|u_t''\|_{\infty[0,3]} \sum_{k=1}^{[1/t]} k + t^2 \sum_{k=[1/t]+1}^{\infty} k \frac{c}{(tk)^3} \\
&\leq c\, t^2 \sum_{k=1}^{[1/t]} k + c\, t^{-1} \sum_{k=[1/t]+1}^{\infty} k^{-2} \leq c.
\end{aligned}$$

This completes the proof of the theorem. □

Remark 4.5. Relations (1.2) and Theorem 1.1 show that $\omega_{2r-1}(f,t)_p$ and $\tilde{\omega}_r^T(f,t)_p$ describe the best trigonometric approximation in terms of big $\mathcal{O}$ rates equally well. However, as we observed earlier, the constants in the two $\mathcal{O}$-estimates can differ considerably. Let us, for simplicity, consider only the case

$r = 2$. Below $c_1, c_2, \ldots$ denote positive absolute constants. A trivial example is given by $f(x) = \sin x$. Then

$$c_1\, t^3 \le \omega_3(f,t)_p \le c_2 t^3, \quad 0 < t \le 1,$$

whereas $\tilde{\omega}_2^T(f,t)_p \equiv 0$.

As another example, let us consider the functions $f_\delta(x) = \sin[(1+\delta)x]$ for $\delta \in (0,1]$. Then for all $\delta \in (0,1]$ we have

$$c_3\, t^3 \le \omega_3(f_\delta,t)_p \le c_4\, t^3, \quad 0 < t \le 1,$$

whereas by properties 6 and 8 we get

$$\tilde{\omega}_2^T(f_\delta,t)_p \le c_5 \left((1+\delta)^3 - 1\right)\, t^3 \le c_6\, \delta\, t^3.$$

Remark 4.6. As for the best constants in the Jackson estimates with the moduli $\omega_{2r-1}(f,t)_p$ and $\tilde{\omega}_r^T(f,t)_p$, respectively, the latter is not better than the former. Chernykh [4] proved for $p = 2$ that

$$\sup_{f \in L_2(\mathbb{T}) \setminus \mathbb{T}_0} \frac{E_{n-1}^T(f)_2}{\omega_m(f, 2\pi/n)_2} = \frac{1}{\sqrt{\binom{2m}{m}}},$$

where $n > m$. This result has quite recently been extended in a certain sense to the other L_p-spaces by Foucart, Kryakin and Shadrin [7]. On the other hand, a result by Babenko [1] implies with $m = 2r - 1$

$$\sup_{f \in L_2(\mathbb{T}) \setminus \mathbb{T}_{r-1}} \frac{E_{n-1}^T(f)_2}{\tilde{\omega}_r^T(f, 2\pi/n)_2} \ge \frac{1}{\sqrt{\displaystyle\max_{h \in [0,2\pi/n]} \sum_{\ell=0}^{m} c_{r,\ell}^2(h)}} \ge \frac{1}{\sqrt{\binom{2m}{m}}}.$$

The second inequality above is derived by Proposition 2.2(iii) and the known identity

$$\sum_{\ell=0}^{m} \binom{m}{\ell}^2 = \binom{2m}{m},$$

which, for example, follows from the Vandermonde convolution formula (e. g. [9, Ch. 1, (3)]). It is reasonable to expect that this relation remains true for $p \neq 2$ as well.

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