A Characterization of Best Algebraic Approximations from Below and from Above in the Multivariate Case

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Abstract

In this paper the constrained K-functionals connected with the best multivariate algebraic approximations from below and from above are characterized in terms of moduli of smoothness. The results are a multivariate generalization of those in [2].

1 Introduction.

We consider measurable real-valued bounded (from below or from above) functions defined in every point of the domain $\Omega = \Pi[-\mathbf{1};\mathbf{1}]$, where

$$\Pi[\mathbf{a};\mathbf{b}] \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^d \;\mid\; x_i \in [\min\{a_i,b_i\},\max\{a_i,b_i\}] \qquad for\;\; every\;\; i=1,\dots,d\right\}.$$

$\mathbb{R}^d$ is considered as a normed vector space with elements $\mathbf{x}=(x_1,\dots,x_d)$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{y}$, $\mathbf{h}$ and norm $\|\mathbf{x}\| = \max\{|x_1|,\dots,|x_d|\}$. Here $\mathbf{1}$ and $-\mathbf{1}$ mean respectively $(1,...,1)$ and $(-1,...,-1)$.

Let X be a measurable subset of Ω. We shall consider the following spaces

$$L_p(X) \stackrel{\text{def}}{=} \left\{f \;\mid\; \|f\|_{p(X)} = \left\{\int_X |f(\mathbf{x})|^p d\mathbf{x}\right\}^{\frac{1}{p}} < \infty\right\},$$

for $p \in [1,\infty)$ ($d\mathbf{x}$ means the Lebesgue measure on X) and

$$L_\infty(X) \stackrel{\text{def}}{=} \left\{f \;\mid\; \|f\|_{\infty(X)} = ess\; sup\,\{|f(\mathbf{x})|\;;\;\mathbf{x}\in X\} < \infty\right\},$$

for $p=\infty$.

α,β are multi-indices. If $\alpha=(\alpha_1,..,\alpha_d)$, $\alpha_s \ge 0$ for any $s=1,...,d$, $|\alpha| = \sum_{i=1}^d \alpha_i$ is the length of α. $\alpha \ge \beta$ means $\alpha_s \ge \beta_s$ for any $s=1,...,d$, $\alpha! = \Pi_{s=1}^d \alpha_s!$ and $\binom{\alpha}{\beta} = \Pi_{s=1}^d \binom{\alpha_s}{\beta_s}$.

Let r be natural. By $W_p^r(X)$ we denote the Sobolev space

$$W_p^r(X) \stackrel{\text{def}}{=} \left\{f \;\mid\; \sum_{|\alpha|=r} \|D^\alpha f\|_{p(X)} < \infty\right\}, \quad where \quad D^\alpha = \prod_{i=1}^d \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\,.$$

For $v \in [-1,1]$, $t > 0$ we set $\psi(t,v) \stackrel{\text{def}}{=} t\sqrt{1-v^2} + t^2$. For $\mathbf{x} \in \Omega$ we denote $\Psi(t,\mathbf{x}) \stackrel{\text{def}}{=} \prod_{s=1}^d \psi(t,x_s)$ and $\Psi^\alpha(t,\mathbf{x}) \stackrel{\text{def}}{=} \prod_{s=1}^d \psi(t,x_s)^{\alpha_s}$. A t neighbourhood of the point $\mathbf{x} \in \Omega$ we define by

$$U(t,\mathbf{x}) \stackrel{\text{def}}{=} \{\mathbf{y} \in \Omega \ | \ |x_s - y_s| \le \psi(t,x_s) \ for\ every \ \ s = 1,...,d\}.$$

Everywhere in this paper c denotes a positive number which may depend on r and d. The c's may differ at each occurrence. If c depends on another parameter we indicate this using indices.

By H_n we denote the set of all algebraic polinomials in $\mathbb{R}^d$ of total degree not greater than n. The best approximations by algebraic polinomials are given by

$$E(f,H_n)_{p(X)} \stackrel{\text{def}}{=} \inf\left\{\|f-Q\|_{p(X)} \ | \ Q \in H_n \right\}$$

and the best approximations from below or from above by algebraic polinomials are given respectively by

$$E^-(f,H_n)_{p(X)} \stackrel{\text{def}}{=} \inf\left\{\|f-Q\|_{p(X)} \ | \ Q \in H_n \ , \ Q \le f\right\} \tag{1.1}$$

and

$$E^+(f,H_n)_{p(X)} \stackrel{\text{def}}{=} \inf\left\{\|f-Q\|_{p(X)} \ | \ Q \in H_n \ , \ Q \ge f\right\}, \tag{1.2}$$

whenever f is bounded from below or from above respectively.

Let $l = \max\left\{\left[\frac{d}{p}\right] + 1 \ , \ r\right\}$ ($[\cdot]-$ integral part). We investigate the K-functionals

$$\begin{aligned} K_r^-(f,t)_p &= K^-\left(f,\Psi(t);L_p,W_p^r,W_p^l\right) \\ &\stackrel{\text{def}}{=} \inf\left\{\|f-g\|_{p(\Omega)} + \sum_{|\alpha|=r,l} \|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega)} \ | \ g \le f \ , \ g \in W_p^l(\Omega)\right\}, \end{aligned} \tag{1.3}$$

$$\begin{aligned} K_r^+(f,t)_p &= K^+\left(f,\Psi(t);L_p,W_p^r,W_p^l\right) \\ &\stackrel{\text{def}}{=} \inf\left\{\|f-g\|_{p(\Omega)} + \sum_{|\alpha|=r,l} \|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega)} \ | \ g \ge f \ , \ g \in W_p^l(\Omega)\right\} \end{aligned} \tag{1.4}$$

and

$$K\left(f,\Psi(t);L_p,W_p^r\right) \stackrel{\text{def}}{=} \inf\left\{\|f-g\|_{p(\Omega)} + \sum_{|\alpha|=r} \|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega)} \ | \ g \in W_p^r(\Omega)\right\}.$$

In [5] we prove the following direct and inverse inequalities for the best constrained approximations in terms of the K-functionals.

Theorem 1.1 *Let $1 \le p \le \infty$, let r and n be natural, $* =$ " $-$ " or " $+$ " and let $f \in L_p(\Omega)$ be bounded from below or from above respectively. Then we have*

$$\begin{array}{ll} (d) & E^*(f, H_{n-1})_{p(\Omega)} \le cK^*\left(f, \Psi(n^{-1}); L_p, W_p^r, W_p^l\right); \\ (i) & K^*\left(f, \Psi(n^{-1}); L_p, W_p^r, W_p^l\right) \le c\left(E^*(f, H_{n-1})_{p(\Omega)} + K\left(f, \Psi(n^{-1}); L_p, W_p^r\right)\right). \end{array}$$

This inequalities are the reason for the investigation in this paper.

The main result of this paper Theorem 1.5 is a characterization for $r = 1$ and $r = 2$ of the K-functional (1.3) in terms of appropriate moduli. As a corollary we give a characterization of the best algebraic approximations from below. Similar results for the K-functional (1.4) and for the best algebraic approximations from above follow as a corollary from $E^+(f) = E^-(-f)$, $K^+(f) = K^-(-f)$ (with one and the same values of the parameters).

The equivalence between the K-functional from below and a characteristic based on local approximations from below by algebraic polinomials we give in

Theorem 1.2 *Let $f \in L_p(\Omega)$ ($p \in [1, \infty]$) be bounded from below and let r be a natural number. Then*

$$K_r^-(f, t)_p \sim \|\Psi(t, \cdot)^{-\frac{1}{p}} E^-(f, H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)} \quad for \quad t \in (0, 1].$$

Remark 1.1. We consider $K_r^-(f, t)_p$ with argument $t \in (0, 1]$ because of Theorem 1.1.

Let $U \subset \mathbb{R}^d$ be a convex body. We set

$$\omega_r(f, U)_p \stackrel{\text{def}}{=} \sup\left\{\|\Delta_{\mathbf{h},U}^r f(\cdot)\|_{p(U)} \;\mid\; \mathbf{h} \in \mathbb{R}^d\right\} \tag{1.5}$$

where

$$\Delta_{\mathbf{h},U}^r f(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} \Delta_{\mathbf{h}}^r f(\mathbf{x}) & if \;\; \mathbf{x}, \mathbf{x} + r\mathbf{h} \in U; \\ 0 & otherwise \end{cases}$$

and

$$\Delta_{\mathbf{h}}^r f(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(\mathbf{x} + i\mathbf{h}).$$

In order to handle the cases of approximations from below for $r = 1$ and $r = 2$ we introduce the following characteristics

$$\tau_r^-(f, U)_p \stackrel{\text{def}}{=} \|\sup\{\tilde{\Delta}_{\mathbf{h},U}^r f(\cdot) \;\mid\; \mathbf{h} \in \mathbb{R}^d\}\|_{p(U)} \tag{1.6}$$

and

$$\tau_r^-(f, \Psi(t))_{p(\Omega)} \stackrel{\text{def}}{=} \|\sup\{\tilde{\Delta}_{\mathbf{h},U(t,\cdot)}^r f(\cdot) \;\mid\; \mathbf{h} \in \mathbb{R}^d\}\|_{p(\Omega)},$$

where

$$\tilde{\Delta}_{\mathbf{h},U}^1 f(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} f(\mathbf{x}) - f(\mathbf{x} + \mathbf{h}) & if \;\; \mathbf{x}, \; \mathbf{x} + \mathbf{h} \in U; \\ 0 & otherwise \end{cases}$$

and

$$\tilde{\Delta}_{\mathbf{h},U}^2 f(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} 2f(\mathbf{x}) - f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x} - \mathbf{h}) & if \;\; \mathbf{x} - \mathbf{h}, \; \mathbf{x} + \mathbf{h} \in U; \\ 0 & otherwise. \end{cases}$$

We use the above characteristics in the following Whitney-type Theorem

Theorem 1.3 *Let $f \in L_p(\Pi)$ $(\Pi = \Pi[\mathbf{a};\mathbf{b}],\ \ p \in [1,\infty])$ is bounded from below. Then*

$$\begin{array}{ll} (I) & E^-(f,H_0)_{p(\Pi)} = \tau_1^-(f,\Pi)_p; \\ (II) & E^-(f,H_1)_{p(\Pi)} \sim \omega_2(f,\Pi)_p + \tau_2^-(f,\Pi)_p. \end{array}$$

Theorem 1.3 is proved in the Section 3.

Let $\Pi = \Pi[\mathbf{a};\mathbf{b}]$ and $\pi = \Pi[\mathbf{c};\mathbf{d}]$ be such that

$$\pi \subseteq \left(\Pi - \frac{\mathbf{a}+\mathbf{b}}{2}\right) \subseteq R.\pi \tag{1.7}$$

for some $R \geq 1$, where for $U \subset \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^d$ and $t > 0$ we denote

$$U + \mathbf{y} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^d \ \mid \ \mathbf{x} - \mathbf{y} \in U\}$$

and

$$tU \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^d \ \mid \ t^{-1}\mathbf{x} \in U\}.$$

We use the following characteristic of f.

$$\tau_r(f,\pi)_{p,p(\Pi)} \stackrel{\text{def}}{=} \left\{\int_\Pi \frac{1}{\mu(\pi)} \int_\pi |\Delta^r_{\mathbf{v},\Pi} f(\mathbf{x})|^p d\mathbf{v} d\mathbf{x}\right\}^{\frac{1}{p}}. \tag{1.8}$$

Here $\mu(V)$ denotes the Lebesgue measure of the measurable set V. A relationship between (1.5) and (1.8) is established in [4, Sec.3]. The statement is

Theorem 1.4 *If (1.7) is satisfied and $f \in L_p(\Omega)$ $(p \in [1,\infty])$ then,*

$$c\tau_r(f,\pi)_{p,p(\Pi)} \leq \omega_r(f,\Pi)_p \leq cR^{d+r}\tau_r(f,\pi)_{p,p(\Pi)}.$$

We set

$$B(t,\mathbf{x}) \stackrel{\text{def}}{=} \left\{\mathbf{y} \in \mathbb{R}^d \ \mid \ |y_s| \leq \psi(t,x_s) \ for\ every \ \ s = 1,...,d\right\}.$$

In this paper we investigate the following averaged modulus of smoothness

$$\tau_r(f,\Psi(t))_{p,p(\Omega)} \stackrel{\text{def}}{=} \left\{\int_\Omega \Psi(t,\mathbf{x})^{-1} \int_{B(t,\mathbf{x})} |\Delta^r_{\mathbf{v},\Omega} f(\mathbf{x})|^p d\mathbf{v} d\mathbf{x}\right\}^{\frac{1}{p}}. \tag{1.9}$$

Using the results from Sections 2, 3, Theorem 1.1 and Theorem 1.4 in Section 4 we give a characterization of the constrained K-functional in terms of appropriate moduli.

Theorem 1.5

$$\begin{array}{ll} K^-(f,\Psi(t),L_p,W_p^1,W_p^{l_1}) \sim \tau_1^-(f,\Psi(t))_{p(\Omega)}, & l_1 = \left[\frac{d}{p}\right] + 1; \\ K^-(f,\Psi(t),L_p,W_p^2,W_p^{l_2}) \sim \tau_2^-(f,\Psi(t))_{p(\Omega)} + \tau_2(f,\Psi(t))_{p,p(\Omega)}, & l_2 = \max\{2, \left[\frac{d}{p}\right] + 1\}. \end{array}$$

Combining the results of Theorem 1.1 and Theorem 1.5 in Section 4 we give a a characterization of best approximation from below in terms of appropriate moduli.

Theorem 1.6

$$E^-(f, H_{n-1})_{p(\Omega)} \le c\tau_1^-(f, \Psi(n^{-1}))_{p(\Omega)};$$

$$E^-(f, H_{n-1})_{p(\Omega)} \le c\{\tau_2^-(f, \Psi(n^{-1}))_{p(\Omega)} + \tau_2(f, \Psi(n^{-1}))_{p,p(\Omega)}\}$$

and for $r = 1$ *and* $r = 2$

$$\tau_r^-(f, \Psi(n^{-1}))_{p(\Omega)} \le c\{E^-(f, H_{n-1})_{p(\Omega)} + \tau_r(f, \Psi(n^{-1}))_{p,p(\Omega)}\}.$$

In order to prove Theorem 1.3(II) we obtain some results for convex functions. Let $U \subset \mathbb{R}^d$ be a convex body. A function $f \ : \ U \to \mathbb{R}$ is called *almost midconvex* if $\sup\{\tilde{\Delta}^2_{\mathbf{h},U} f(\mathbf{x}) \ | \ \mathbf{h} \in \mathbb{R}^d\} = 0$ holds for every $\mathbf{x} \in U$ except a subset of U with a measure zero. As a corollary from the results in the Section 3 we get

Theorem 1.7 *If a function* $f \ : \ \Pi[\mathbf{a}; \mathbf{b}] \to \mathbb{R}$ *is almost midconvex then* f *is equal almost everywhere to a convex function* g *and* $f \ge g$*.*

2 A characterization of (1.3) in terms of best algebraic local approximation from below.

Here we use methots which are based on ideas of [2], [3] and [5] and prove Theorem 1.2. We start with some notations.

Let N be a fixed natural number. We set

$$\mathbb{Z} = \{0,\ 1,\ ...,\ N-1\}^d \ ; \ \mathbb{Z}' = \{0,\ 1,\ ...,\ N\}^d \ ; \ \mathbb{E} = \{0,\ 1\}^d;$$

$$z_k = cos(\pi - \frac{k\pi}{N}),\ k = 0, 1, ..., N,\ z_{-1} = z_0 = -1,\ \ z_{N+1} = z_N = 1.$$

For every $\mathbf{j} = (j_1,\ j_2,\ ...,\ j_d) \in \mathbb{Z}$ we denote

$$\mathbf{\Omega_j} = [z_{j_1},\ z_{j_1+1}] \times \ ... \ \times [z_{j_d},\ z_{j_d+1}]$$

and for every $\mathbf{j} \in \mathbb{Z}'$ we denote

$$\mathbf{\Omega'_j} = [z_{j_1-1},\ z_{j_1+1}] \times \ ... \ \times [z_{j_d-1},\ z_{j_d+1}].$$

We set $\mu(v) = \int_0^v e^{\frac{-1}{u-u^2}} du / \int_0^1 e^{\frac{-1}{u-u^2}} du$ for $0 < v < 1$, $\mu(v) = 0$ for $v \le 0$ and $\mu(v) = 1$ for $v \ge 1$. Therefore $\mu \in C^\infty(\mathbb{R})$ and we define

$$\mu_0(v) \stackrel{\text{def}}{=} 1 - \mu((v - z_0)/(z_1 - z_0));$$
$$\mu_s(v) \stackrel{\text{def}}{=} \mu((v - z_{s-1})/(z_s - z_{s-1}))(1 - \mu((v - z_s)/(z_{s+1} - z_s))) \quad for \quad s = 1,\ 2, ...,\ N-1;$$
$$\mu_N(v) \stackrel{\text{def}}{=} \mu((v - z_{N-1})/(z_N - z_{N-1})).$$

Finaly for every $\mathbf{j} \in \mathbb{Z}'$ we set $\mu_{\mathbf{j}}(\mathbf{x}) = \Pi_{s=1}^d \mu_{j_s}(x_s)$. Therefore for every $\mathbf{x} \in \Omega$ we have

$$0 \le \mu_{\mathbf{j}}(\mathbf{x}) \le 1; \quad \mu_{\mathbf{j}}(\mathbf{x}) = 0 \ \ if \ \ \mathbf{x} \notin \Omega'_{\mathbf{j}}; \tag{2.1}$$

$$\sum_{\mathbf{j} \in \mathbb{Z}'} \mu_{\mathbf{j}}(\mathbf{x}) = 1. \tag{2.2}$$

In the statements below we collect some properties of the above quantities. Let $0 < t \le \frac{1}{2}$ and $N = \left[\frac{2\pi}{t}\right] + 1$. Then we have

$$\Psi(t, \mathbf{x}) \le meas(U(t, \mathbf{x})) \le 2^d \Psi(t, \mathbf{x}); \tag{2.3}$$

$$\Psi(t, \mathbf{x}) \sim \Psi(t, \mathbf{y}) \ \ for \ \ every \ \mathbf{y} \in U(t, \mathbf{x}); \tag{2.4}$$

$$\Psi(t, \mathbf{x}) \sim \Psi(t, \mathbf{x} + \mathbf{y}) \ \ for \ \ every \ \mathbf{y} \in B(t, \mathbf{x}); \tag{2.5}$$

$$c\Psi(t, \mathbf{x}) \le meas(\Omega'_{\mathbf{j}}) \le c\Psi(t, \mathbf{y}) \ \ for \ \ every \ \ x,\ y \in \Omega'_{\mathbf{j}}; \tag{2.6}$$

$$\Omega'_{\mathbf{j}} \subset U(t, \mathbf{x}) \ \ for \ \ any \ \ \mathbf{x} \in \Omega'_{\mathbf{j}}. \tag{2.7}$$

The inequalities (2.3), (2.4), (2.6) and (2.7) are proved in [3]. (2.5) follows from (2.3), (2.4) and definition of $B(t, \mathbf{x})$.

We prove first the following

Lemma 2.1 *Let $0 < t \le \frac{1}{2}$. Then for every $f \in L_p(\Omega)$ we have*

$$\|\Psi(t, \cdot)^{-\frac{1}{p}} E^-(f, H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)} \le c K_r^-(f, t)_p; \tag{2.8}$$

$$K_r^-(f, t)_p \le c \|\Psi(t, \cdot)^{-\frac{1}{p}} E^-(f, H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)}. \tag{2.9}$$

Proof. Let us begin with the proof of (2.9). We set

$$N = \left[\frac{2\pi}{t}\right] + 1 \tag{2.10}$$

and use the notation for $\Omega_{\mathbf{j}}$, $\Omega'_{\mathbf{j}}$ and $\mu_{\mathbf{j}}$ from the beginning of Section 2. We denote by $Q_{\mathbf{j}} \in H_{r-1}$ the polynomial of best algebraic L_p approximation from below of degree $r-1$ to f in $\Omega'_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}'$. We set

$$g(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{Z}'} \mu_{\mathbf{j}}(\mathbf{x}) Q_{\mathbf{j}}(\mathbf{x}). \tag{2.11}$$

From (2.10), (2.11), (2.3), (2.6) and (2.7) we obtain

$$
\begin{aligned}
(2.12) \qquad \|f-g\|^p_{p(\Omega)} &= \|\sum_{\mathbf{j}\in\mathbb{Z}'}\mu_{\mathbf{j}}(f-Q_{\mathbf{j}})\|^p_{p(\Omega)} \\
&\le c\sum_{\mathbf{j}\in\mathbb{Z}'}\int_{\Omega'_{\mathbf{j}}}|f(\mathbf{x})-Q_{\mathbf{j}}(\mathbf{x})|^p d\mathbf{x} \\
&= c\sum_{\mathbf{j}\in\mathbb{Z}'}E^-(f,H_{r-1})^p_{p(\Omega'_{\mathbf{j}})} \\
&= c\sum_{\mathbf{j}\in\mathbb{Z}'}meas(\Omega'_{\mathbf{j}})^{-1}\int_{\Omega'_{\mathbf{j}}}E^-(f,H_{r-1})^p_{p(\Omega'_{\mathbf{j}})}d\mathbf{x} \\
&\le c\sum_{\mathbf{j}\in\mathbb{Z}'}meas(\Omega'_{\mathbf{j}})^{-1}\int_{\Omega'_{\mathbf{j}}}E^-(f,H_{r-1})^p_{p(U(t,\mathbf{x}))}d\mathbf{x} \\
&\le c\sum_{\mathbf{j}\in\mathbb{Z}'}\int_{\Omega'_{\mathbf{j}}}(\Psi(t,\mathbf{x}))^{-1}E^-(f,H_{r-1})^p_{p(U(t,\mathbf{x}))}d\mathbf{x} \\
&\le c\int_{\Omega}(\Psi(t,\mathbf{x}))^{-1}E^-(f,H_{r-1})^p_{p(U(t,\mathbf{x}))}d\mathbf{x} \\
&\le c\|\Psi(t,\cdot)^{-\frac{1}{p}}E^-(f,H_{r-1})_{p(U(t,\cdot))}\|^p_{p(\Omega)}.
\end{aligned}
$$

Fix α , $|\alpha|=r\ or\ |\alpha|=l$. Let $\mathbf{x}\in\Omega_{\mathbf{j}}$, $\mathbf{j}\in\mathbb{Z}$. From the definitions of $\mu(\mathbf{x})$, $Q_{\mathbf{j}}(\mathbf{x})$ and $g(\mathbf{x})$ we have

$$g(\mathbf{x})=Q_{\mathbf{j}}(\mathbf{x})+\sum_{\epsilon\in\mathbb{E}}\mu_{\mathbf{j}+\epsilon}(\mathbf{x})\left(Q_{\mathbf{j}+\epsilon}(\mathbf{x})-Q_{\mathbf{j}}(\mathbf{x})\right),\ where$$

$$\mu_{\mathbf{j}+\epsilon}(\mathbf{x})=\prod_{s;\,\epsilon_s=1}\mu\left(\frac{x_s-z_{j_s}}{z_{j_s+1}-z_{j_s}}\right)\cdot\prod_{s;\,\epsilon_s=0}\mu\left(1-\left(\frac{x_s-z_{j_s}}{z_{j_s+1}-z_{j_s}}\right)\right).$$

and then from the last equality and $D^\alpha Q_{\mathbf{j}}=0$, it follows that

$$D^\alpha g(\mathbf{x})=\sum_{\epsilon\in\mathbb{E}}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}\mu_{\mathbf{j}+\epsilon}(\mathbf{x})D^\beta\left(Q_{\mathbf{j}+\epsilon}(\mathbf{x})-Q_{\mathbf{j}}(\mathbf{x})\right)$$

Now using (2.6), (2.7), the definitions of $\mu_{\mathbf{j}}$, $Q_{\mathbf{j}}$ and $\mathbb{E}$ and Markov's inequality ($(b-a)^i\|g^{(i)}\|_{p[a,b]}\le c(r)\|g\|_{p[a,b]}$ for $g\in H_r$) we have

$$
\begin{aligned}
\|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega_{\mathbf{j}})} &\le c\Psi^\alpha(t,z_{\mathbf{j}})\|D^\alpha g\|_{p(\Omega_{\mathbf{j}})} \\
&\le c\Psi^\alpha(t,z_{\mathbf{j}})\sum_{\epsilon\in\mathbb{E}}\sum_{\mathbf{0}\le\beta\le\alpha}\binom{\alpha}{\beta}\|D^{\alpha-\beta}\mu_{\mathbf{j}+\epsilon}\|_{\infty(\Omega_{\mathbf{j}})}\|D^\beta\left(Q_{\mathbf{j}+\epsilon}-Q_{\mathbf{j}}\right)\|_{p(\Omega_{\mathbf{j}})} \\
&\le c\Psi^\alpha(t,z_{\mathbf{j}})\sum_{\epsilon\in\mathbb{E}}\sum_{\mathbf{0}\le\beta\le\alpha}\prod_{s=1}^{d}\frac{\|\mu^{(|\alpha-\beta|)}\|_{\infty[0,1]}}{|z_{j_s+1}-z_{j_s}|^{\alpha_s-\beta_s}}\|D^\beta\left(Q_{\mathbf{j}+\epsilon}-Q_{\mathbf{j}}\right)\|_{p(\Omega_{\mathbf{j}})} \\
&\le c\sum_{\epsilon\in\mathbb{E}}\sum_{\mathbf{0}\le\beta\le\alpha}\prod_{s=1}^{d}|z_{j_s+1}-z_{j_s}|^{\beta_s}\|D^\beta\left(Q_{\mathbf{j}+\epsilon}-Q_{\mathbf{j}}\right)\|_{p(\Omega_{\mathbf{j}})} \\
&\le c\sum_{\epsilon\in\mathbb{E}}\|Q_{\mathbf{j}+\epsilon}-Q_{\mathbf{j}}\|_{p(\Omega_{\mathbf{j}})}
\end{aligned}
$$

$$\begin{aligned} &\leq \; c\sum_{\epsilon\in\mathbb{E}}\Big(\|f-Q_{\mathbf{j}+\epsilon}\|_{p(\Omega_{\mathbf{j}})}+\|f-Q_{\mathbf{j}}\|_{p(\Omega_{\mathbf{j}})}\Big)\\ &\leq \; cE^-(f,H_{r-1})^p_{p(\Omega'_{\mathbf{j}})}. \end{aligned}$$

Hence

$$\begin{aligned} \|\Psi^\alpha(t)D^\alpha g\|^p_{p(\Omega)} &\leq c\sum_{\mathbf{j}\in\mathbb{Z}'}E^-(f,H_{r-1})^p_{p(\Omega'_{\mathbf{j}})}\\ &= c\sum_{\mathbf{j}\in\mathbb{Z}'}meas(\Omega'_{\mathbf{j}})^{-1}\int_{\Omega'_{\mathbf{j}}}E^-(f,H_{r-1})^p_{p(\Omega'_{\mathbf{j}})}d\mathbf{x}\\ &\leq c\sum_{\mathbf{j}\in\mathbb{Z}'}meas(\Omega'_{\mathbf{j}})^{-1}\int_{\Omega'_{\mathbf{j}}}E^-(f,H_{r-1})^p_{p(U(t,\mathbf{x}))}d\mathbf{x}\\ &\leq c\sum_{\mathbf{j}\in\mathbb{Z}'}\int_{\Omega'_{\mathbf{j}}}(\Psi(t,\mathbf{x}))^{-1}E^-(f,H_{r-1})^p_{p(U(t,\mathbf{x}))}d\mathbf{x}\\ &\leq c\int_{\Omega}(\Psi(t,\mathbf{x}))^{-\frac{1}{p}}E^-(f,H_{r-1})^p_{p(U(t,\mathbf{x}))}d\mathbf{x}\\ &\leq c\|\Psi(t,\cdot)^{-\frac{1}{p}}E^-(f,H_{r-1})_{p(U(t,\cdot))}\|^p_{p(\Omega)}. \end{aligned} \tag{2.13}$$

In this way (2.9) follows from (1.3), (2.12) and (2.13).

We turn our attention to (2.8).

Let $\alpha=(\alpha_1,..,\alpha_d)$, $|\alpha|=r$ be multi-index and $\mathbf{z}=(z_1,...,z_d)\in\mathbb{R}^d$. We define

$$|\mathbf{z}^\alpha|=\Pi_{i=1}^d|z_i|^{\alpha_i}$$

Let $\Pi=\Pi[\mathbf{a};\mathbf{b}]$ and let $g\in W^l_p(\Pi)$. As a corollary from Theorem 2 and Theorem 1 in [3] we get

$$E^-(g,H_{r-1})_{p(\Pi)}\leq c\sum_{|\alpha|=r,l}|(\mathbf{b}-\mathbf{a})^\alpha|\;\;\|D^\alpha g\|_{p(\Pi)}. \tag{2.14}$$

Let g be any function in $W^l_p(\Omega)$, $g(\mathbf{x})\leq f(\mathbf{x})$, $\mathbf{x}\in\Omega$. Then we have (note that $Q\leq g$ implies $Q\leq f$)

$$E^-(f,H_{r-1})_{p(U(t,\mathbf{x}))}\leq E^-(g,H_{r-1})_{p(U(t,\mathbf{x}))}+\|f-g\|_{p(U(t,\mathbf{x}))}$$

and hence

$$\begin{aligned} &\|\Psi(t,\cdot)^{-\frac{1}{p}}E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)}\\ &\leq \|\Psi(t,\cdot)^{-\frac{1}{p}}E^-(g,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)}+\|\Psi(t,\cdot)^{-\frac{1}{p}}\|f-g\|_{p(U(t,\cdot))}\|_{p(\Omega)}. \end{aligned} \tag{2.15}$$

Using (2.14), (2.3) and (2.4) we obtain

$$\begin{aligned} E^-(g,H_{r-1})_{p(U(t,\mathbf{x})))} &\leq c\sum_{|\alpha|=r,l}\Psi^\alpha(t,\mathbf{x})\|D^\alpha g\|_{p(U(t,\mathbf{x}))}\\ &\leq c\sum_{|\alpha|=r,l}\|\Psi^\alpha(t,\cdot)D^\alpha g\|_{p(U(t,\mathbf{x}))}. \end{aligned} \tag{2.16}$$

From Lemma 4 in [3] we have that

$$\|\Psi^{-\frac{1}{p}}(t,\cdot)\|G\|_{p(U(t,\cdot))}\|_{p(\Omega)} \le c\|G\|_{p(\Omega)} \tag{2.17}$$

for $G \in L_p(\Omega)$ and $t \in (0, \frac{1}{2}]$. Then from (2.16) and (2.17) we get

$$\|\Psi^{-\frac{1}{p}}(t,\cdot)\|f-g\|_{p(U(t,\cdot))}\|_{p(\Omega)} \le c\|f-g\|_{p(\Omega)};$$

$$\|\Psi^{-\frac{1}{p}}(t,\cdot)\|E^-(g,H_{r-1})\|_{p(U(t,\cdot))}\|_{p(\Omega)} \le c\sum_{|\alpha|=r,l}\|\Psi^{\alpha}(t,\cdot)D^{\alpha}g\|_{p(\Omega)}.$$

Hence using (2.15) we get

$$\|\Psi^{-\frac{1}{p}}(t,\cdot)\|E^-(f,H_{r-1})\|_{p(U(t,\cdot))}\|_{p(\Omega)} \le c\left\{\|f-g\|_{p(\Omega)} + \sum_{|\alpha|=r,l}\|\Psi^{\alpha}(t)D^{\alpha}g\|_{p(\Omega)}\right\}.$$

Taking an infimum on all $g \in W_p^r(\Omega)$, $g \le f$ in the above inequality we prove (2.8). □

Proof of Theorem 1.2. We have to investigate only the case $t \in (\frac{1}{2}, 1]$, because for $t \in (0, \frac{1}{2}]$ Theorem 1.2 is equal to Lemma 2.1. Let $t \in (\frac{1}{2}, 1]$. Then from the definitions of $\Psi(t, \mathbf{x})$ and $U(t, \mathbf{x})$ it follows that

$$\begin{aligned}&\|\Psi^{-\frac{1}{p}}(t,\cdot)E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)} \le 4^d E^-(f,H_{r-1})_{p(\Omega)}\\ &\le c\|\Psi^{-\frac{1}{p}}(\frac{1}{2},\cdot)E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)} \le c\|\Psi^{-\frac{1}{p}}(t,\cdot)E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)}.\end{aligned}$$

Hence from Lemma 2.1(2.8) with $t = \frac{1}{2}$ and the monotonicity of the K-functional (1.3) with respect to t we get

$$\begin{aligned}\|\Psi^{-\frac{1}{p}}(t,\cdot)E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)} &\le cK_r^-(f,\frac{1}{2})_p \le cK_r^-(f,t)_p\ ;\\ K_r^-(f,t)_p \le cE^-(f,H_{r-1})_{p(\Omega)} &\le c\|\Psi^{-\frac{1}{p}}(t,\cdot)E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)}.\end{aligned}$$

□

3 Whitney-type theorems for best approximations from below.

We make use of some properties of the moduli which follows immediately from the definition

$$\tau_r^-(f,U)_p \le r\ \|f\|_{p(U)}\ \textit{for}\ r=1\ ,\ 2\quad \textit{if}\ \ f(\mathbf{x}) \ge 0\ \ \textit{for every}\ \mathbf{x} \in U; \tag{3.1}$$

$$\tau_2^-(f,U)_p = 0\ \ \textit{if}\ \ f\ \ \textit{is convex on}\ \ U\ ; \tag{3.2}$$

$$\tau_r^-(f+g,U)_p \le \tau_r^-(f,U)_p\ +\ \tau_r^-(g,U)_p\quad \textit{for}\ r=1\ ,\ 2; \tag{3.3}$$

$$\tau_2^-(f_+,U)_p \le \tau_2^-(f,U)_p,\ \textit{where}\ f(\mathbf{x})_+ \stackrel{\text{def}}{=} \begin{cases} f(\mathbf{x}) & \textit{if}\ \ f(\mathbf{x}) \ge 0;\\ 0 & \textit{otherwise.}\end{cases} \tag{3.4}$$

Remark 3.1. In (3.1) and (3.3) we use $\sup\{\tilde{\Delta}^r_{\mathbf{h},U} f(\mathbf{x}) \mid \mathbf{h} \in \mathbb{R}^d\} \geq \tilde{\Delta}^r_{\mathbf{0},U} f(\mathbf{x}) = 0$.
Remark 3.2. $\tau_r^-(f-g,U)_p \leq \tau_r^-(f,U)_p + \tau_r^-(g,U)_p$ is not true in general. For example $d=1$, $r=2$, $U=[-1,1]$, $f(x)=const$ and $g(x)=x^2$.
Remark 3.3. In (3.4) we use that if $f(\mathbf{x}) \geq 0$ then

$$\begin{aligned}\sup\{\tilde{\Delta}^2_{\mathbf{h},U} f(\mathbf{x})_+ \mid \mathbf{h} \in \mathbb{R}^d\} &= \sup\{2f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})_+ - f(\mathbf{x}-\mathbf{h})_+ \mid \mathbf{h} \in \mathbb{R}^d\} \\ &\leq \sup\{\tilde{\Delta}^2_{\mathbf{h},U} f(\mathbf{x}) \mid \mathbf{h} \in \mathbb{R}^d\}\end{aligned}$$

and if $f(\mathbf{x}) < 0$ then $\sup\{\tilde{\Delta}^2_{\mathbf{h},U} f(\mathbf{x})_+ \mid \mathbf{h} \in \mathbb{R}^d\} = 0 \leq \sup\{\tilde{\Delta}^2_{\mathbf{h},U} f(\mathbf{x}) \mid \mathbf{h} \in \mathbb{R}^d\}$.

Proof of Theorem 1.3(I). The statement (I) of Theorem 1.3 is similar to Theorem 4.1 from [2] and the proof is the same. Let $M = \inf\{f(\mathbf{y}) \mid \mathbf{y} \in \Pi = \Pi[\mathbf{a};\mathbf{b}]\}$. Then $E^-(f,H_0)_{p(\Pi)} = \|f - M\|_{p(\Pi)}$ and for every $\mathbf{x} \in \Pi$ we have

$$\begin{aligned} f(\mathbf{x}) - M &= f(\mathbf{x}) - \inf\{f(\mathbf{y}) \mid \mathbf{y} \in \Pi\} \\ &= \sup\{f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h}) \mid \mathbf{x}+\mathbf{h} \in \Pi\} \\ &= \sup\{\tilde{\Delta}^1_{\mathbf{h},\Pi} f(\mathbf{x}) \mid \mathbf{h} \in \mathbb{R}^d\}. \end{aligned}$$

Taking L_p norm in this inequality we prove the lemma. □

Now we turn our attention to the case $r = 2$(Theorem 1.3 (II)). We start with some lemmas which are conected with the best multivariate algebraic approximations from below of convex functions.

Lemma 3.1 *Let $E \subset \Pi[\mathbf{a};\mathbf{b}] \subset \mathbb{R}^d$ be an open convex body with a measure $\mu(E) < \frac{1}{2^d d!}\mu(\Pi[\mathbf{a};\mathbf{b}])$. Then there are $k \in \{1,...,d\}$ and measurable subsets E_{k-} and E_{k+} and for every $\mathbf{x} \in E$ there exist $s(\mathbf{x}) \in \partial E$ and $\mathbf{y}(\mathbf{x}) \in \Pi[\mathbf{a};\mathbf{b}] \setminus (E \cup \partial E)$, such that*

1) $E = E_{k-} \cup E_{k+}$;

2) If $\mathbf{x} \in E_{k*}$ ($* = +$ or $-$), $\mathbf{y}(\mathbf{x}) - \mathbf{s}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) - \mathbf{x} = t_{\mathbf{x}} e_k$, where e_k is the “k-th“ unit coordinate vector and $sign(t_{\mathbf{x}}) = *$.

Proof. Let $\mathbf{x} \in E$ and $k \in \{1,...,d\}$. We define

$$E_k(\mathbf{x}) \stackrel{\text{def}}{=} \{\mathbf{z} \in E \mid z_i = x_i \ \forall i = 1,...,d\ ,\ i \neq k\ \},$$

$$m^-_{k,E}(\mathbf{x}) \stackrel{\text{def}}{=} \inf\{z_k \mid \mathbf{z} \in E_k(\mathbf{x})\ \}$$

and

$$m^+_{k,E}(\mathbf{x}) \stackrel{\text{def}}{=} \sup\{z_k \mid \mathbf{z} \in E_k(\mathbf{x})\ \}.$$

From $\mu(E) < \frac{1}{2^d d!}\mu(\Pi[\mathbf{a};\mathbf{b}])$ and convexity of E we have that there exists $k \in \{1,...,d\}$ such that $m^+_{k,E}(\mathbf{x}) - m^-_{k,E}(\mathbf{x}) < \frac{|b_k - a_k|}{2}$ for every $\mathbf{x} \in E$. (If we assume that for every $k \in \{1,...,d\}$

there exist $\mathbf{x}(k) \in E$ such that $m^+_{k,E}(\mathbf{x}(k)) - m^-_{k,E}(\mathbf{x}(k)) \geq \frac{|b_k - a_k|}{2}$ then from convexity of E we have that $\mu(E) \geq \frac{1}{2^d d!}\mu(\Pi[\mathbf{a};\mathbf{b}])$.) Thus we reduce the problem for $E \subset \mathbb{R}^d$ to the problem for $E_k(\mathbf{x}) \subset \mathbb{R}^1$. We define

$$c_{k,E}(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} m^+_{k,E}(\mathbf{x}) & if \ \ \frac{b_k+a_k}{2} \leq m^-_{k,E}(\mathbf{x}) \\ m^-_{k,E}(\mathbf{x}) + m^+_{k,E}(\mathbf{x}) - \frac{b_k+a_k}{2} & if \ \ \frac{b_k+a_k}{2} \in \left(m^-_{k,E}(\mathbf{x}), m^+_{k,E}(\mathbf{x})\right) \\ m^-_{k,E}(\mathbf{x}) & if \ \ \frac{b_k+a_k}{2} \geq m^+_{k,E}(\mathbf{x}). \end{cases}$$

We set $E_{k-} = \left\{\mathbf{z} \in E \mid z_k \in (m^-_{k,E}(\mathbf{z}), c_{k,E}(\mathbf{z})\right]$, $E_{k+} = \left\{\mathbf{z} \in E \mid z_k \in (c_{k,E}(\mathbf{z}), m^+_{k,E}(\mathbf{z})\right)$ and let $s(\mathbf{x}) = (s(\mathbf{x})_1, ..., s(\mathbf{x})_d)$ be such that

$$s(\mathbf{x})_i \stackrel{\text{def}}{=} \begin{cases} x_i & if \ \ i \neq k \\ m^*_{k,E}(\mathbf{x}) & if \ \ i = k \ \ and \ \ \mathbf{x} \in E_{k*}, \quad * = + \ or \ - \ . \end{cases}$$

The funcions $m^-_{k,E}(\mathbf{x})$ and $m^+_{k,E}(\mathbf{x})$ are continuos because they are face functions of the convex body E. Then from the construction the subsets E_{k-} and E_{k+} have continuos boundary and then they are measurable. Also from the construction of the subsets E_{k-} and E_{k+} we have $E = E_{k-} \cup E_{k+}$, $\mathbf{y}(\mathbf{x}) = 2\mathbf{s}(\mathbf{x}) - \mathbf{x} \in \Pi[\mathbf{a};\mathbf{b}] \setminus (E \cup \partial E)$ and if $\mathbf{x} \in E_{k*}$ $(* = \ + \ or \ - \ )$ then $\mathbf{y}(\mathbf{x}) - \mathbf{s}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) - \mathbf{x} = t_{\mathbf{x}} e_k$, where e_k is the "k-th" unit coordinate vector and $sign(t_{\mathbf{x}}) = *$.

□

Lemma 3.2 *Let f be convex on $\Pi[\mathbf{0};\mathbf{a}]$, $f(\mathbf{x}) \geq 0$, $f(\mathbf{0}) = 0$ and $p \in [1, \infty)$. Then*

$$\|f\|_{p(\Pi[\mathbf{0};\mathbf{a}])} \leq cE(f, H_0)_{p(\Pi[\mathbf{0};\mathbf{a}])}.$$

Proof. Let M be such that $E^-(f, H_0)_p = \|f - M\|_p$. If $M = 0$ (which is possible for example when $p = 1$ and f vanishes in a set E with a measure $\mu(E) \geq \frac{1}{2}\mu(\Pi[\mathbf{0};\mathbf{a}])$), then the statement of Lemma 3.2 holds as an equality with constant $c = 1$. In the other cases (when $M > 0$) we set $E_* = \{\mathbf{x} \in \Pi[\mathbf{0};\mathbf{a}] \mid sign(f(\mathbf{x}) - M) = *\}$, where $* =$ " $-$ " *or* " $+$ " .

Let $E_0 = \partial E_-$. For $\mathbf{x} \in E_-$ we define $g(\mathbf{x}) \stackrel{\text{def}}{=} t_{\mathbf{x}} M$, where $\mathbf{x} = t_{\mathbf{x}}\mathbf{y}$ and $\mathbf{y} \in E_0$. Since $\int_{E_-}(M - f)^p \geq \int_{E_-}(M - g)^p$ and

$$\mu\left\{\mathbf{x} \in E_- \mid (1 - t_{\mathbf{x}})^p \geq \theta\right\} = \mu\left\{\mathbf{x} \in E_- \mid t_{\mathbf{x}} \leq (1 - \theta^{\frac{1}{p}})\right\} = \mu(E_-)(1 - \theta^{\frac{1}{p}})^d,$$

we have

$$\begin{aligned} (3.5) \qquad \int_{E_-}(M - f(\mathbf{x}))^p d\mathbf{x} &\geq \int_{E_-}(M - g(\mathbf{x}))^p d\mathbf{x} \\ &= M^p \int_{E_-}(1 - t_{\mathbf{x}})^p d\mathbf{x} \\ &= M^p \int_{E_-}\int_0^{(1-t_{\mathbf{x}})^p} 1 d\theta d\mathbf{x} \\ &= M^p \int_0^1 \int_{(1-\theta^{\frac{1}{p}})^d E_-} 1 d\mathbf{x} d\theta \end{aligned}$$

$$
\begin{aligned}
&= M^p \mu(E_-) \int_0^1 (1-\theta^{\frac{1}{p}})^d d\theta \\
&= M^p \mu(E_-) p \int_0^1 t^d (1-t)^{p-1} dt \\
&= \binom{p+d}{d}^{-1} \int_{E_-} M^p.
\end{aligned}
$$

From (3.5) and the trivial inequality $\int_{E_-} f^p \le \int_{E_-} M^p$ we get

$$
\int_{E_-} f^p \le \binom{p+d}{d} \int_{E_-} (M-f)^p. \tag{3.6}
$$

$f - M$ and M are positive numbers in E_+. Then the convexity of x^p gives

$$
\int_{E_+} f^p \le 2^{p-1} \int_{E_+} (f-M)^p + 2^{p-1} \int_{E_+} M^p. \tag{3.7}
$$

If we assume that $\mu(E_-) < \frac{1}{2^d d!}\mu(\Pi[\mathbf{0};\mathbf{a}])$ then from Lemma 3.1 (with $E = E_-$) and the convexity of f we have that $M - f(\mathbf{x}) \le f(\mathbf{y}(\mathbf{x})) - M$ ($f(\mathbf{s}(\mathbf{x})) = M$) for any $\mathbf{x} \in E_-$. Take the power $p-1$ in the both sides of the last inequality and integrating on $\mathbf{x} \in E_-$ we obtain

$$
\int_{E_-} (M - f(\mathbf{x}))^{p-1} d\mathbf{x} \le \int_{E_-} (f(\mathbf{y}(\mathbf{x})) - M)^{p-1} d\mathbf{x} < \int_{E_+} (f(\mathbf{y}) - M)^{p-1} d\mathbf{y}.
$$

But this is a contradiction, because from the characterization of the element of best approximation (see [7]) we have

$$
\int_{\Pi[\mathbf{0};\mathbf{a}]} (f(\mathbf{x}) - M)^{p-1} sign(f(\mathbf{x}) - M) d\mathbf{x} = 0.
$$

Therefore $\mu(E_-) \ge \frac{1}{2^d d!}\mu(\Pi[\mathbf{0};\mathbf{a}])$ i.e. $\mu(E_+) \le (2^d d! - 1)\mu(E_-)$.

Then using the last one, (3.5), (3.6) and (3.7) we obtain

$$
\int_{E_+} f^p \le 2^{p-1} \int_{E_+} (f-M)^p + 2^{p-1} \binom{p+d}{d} (2^d d! - 1) \int_{E_-} (M-f)^p. \tag{3.8}
$$

Now from (3.6) and (3.8) it follows

$$
\|f\|_{p(\Pi[\mathbf{0};\mathbf{a}])} \le c \|f - M\|_{p(\Pi[\mathbf{0};\mathbf{a}])}.
$$

□

Lemma 3.3 *If f is convex in $\Pi[\mathbf{a};\mathbf{b}]$ then $E^-(f, H_1)_{p(\Pi[\mathbf{a};\mathbf{b}])} \le c\, E(f, H_1)_{p(\Pi[\mathbf{a};\mathbf{b}])}$.*

Proof. Without loss of generality we may assume that $f(\mathbf{y}) = \liminf_{\mathbf{x}\to\mathbf{y}} f(\mathbf{x})$ for every boundary point $\mathbf{y}$ of $\Pi[\mathbf{a};\mathbf{b}]$. Let $Q(\mathbf{x})$ be the first degree polinomial of best L_p approximation to f. The function $f-Q$ is convex and let its minimum be achieved (from the above assumtion) at the point $\mathbf{u} \in \Pi[\mathbf{a};\mathbf{b}]$. Applying Lemma 3.2 to the convex function

$$g(\mathbf{x}) = f(\mathbf{x}) - Q(\mathbf{x}) - (f(\mathbf{u}) - Q(\mathbf{u})),$$

separately on the parallelepipeds $\Pi[\mathbf{u};\alpha\mathbf{a}+(\mathbf{1}-\alpha)\mathbf{b}]$, where $\alpha = (\alpha_1 \dots \alpha_d) \in \{0,1\}^d$ we have

$$\|g\|_{p(\Pi[\mathbf{u};\alpha\mathbf{a}+(\mathbf{1}-\alpha)\mathbf{b}])} \le c\, E(g,H_0)_{p(\Pi[\mathbf{u};\alpha\mathbf{a}+(\mathbf{1}-\alpha)\mathbf{b}])} \le c\|g+(f(\mathbf{u})-Q(\mathbf{u}))\|_{p(\Pi[\mathbf{u};\alpha\mathbf{a}+(\mathbf{1}-\alpha)\mathbf{b}])}$$

for every $\alpha \in \{0,1\}^d$.

Now adding the above inequalities raised to the power p and then take the power $\frac{1}{p}$ of the sum we obtain

$$\|f-Q-(f(\mathbf{u})-Q(\mathbf{u}))\|_{p(\Pi[\mathbf{a};\mathbf{b}])} \le c\|f-Q\|_{p(\Pi[\mathbf{a};\mathbf{b}])} = c\, E(f,H_1)_{p(\Pi[\mathbf{a};\mathbf{b}])}.$$

But $f(\mathbf{x}) \ge Q(\mathbf{x}) + f(\mathbf{u}) - Q(\mathbf{u})$ and $Q(x) + f(\mathbf{u}) - Q(\mathbf{u}) \in H_1$. Then

$$E^-(f,H_1)_{p(\Pi[\mathbf{a};\mathbf{b}])} \le \|f-Q-(f(\mathbf{u})-Q(\mathbf{u}))\|_{p(\Pi[\mathbf{a};\mathbf{b}])} \le c\, E(f,H_1)_{p(\Pi[\mathbf{a};\mathbf{b}])}.$$

□

In order to get the result for the approximations from below by linear functions we need some statements which are more complificated than in case $d=1$.

Let $A = \mathbf{A}_1 \dots \mathbf{A}_{d+1}$ be d-dimensional simplex. If $\mathbf{x} \in A$, then $\mathbf{x} = \sum_{i=1}^{d+1} \alpha_i(\mathbf{x})\mathbf{A}_i$, where $\alpha_i(\mathbf{x}) \ge 0$ and $\sum_{i=1}^{d+1} \alpha_i(\mathbf{x}) = 1$. $\alpha_1(\mathbf{x}), ..., \alpha_{d+1}(\mathbf{x})$ are usually called *barycentric coordinates* of $\mathbf{x}$ with respect to $\mathbf{A}_1, \dots, \mathbf{A}_{d+1}$.

For $n > d$ and $i = 1 \dots n-1$ we consider the following subsets of A

$$M^n_{k,i} = \{\mathbf{x} \in A \mid \alpha_k(\mathbf{x}) \ge \frac{i}{n} \}. \tag{3.9}$$

Let $i_k(\mathbf{x}) = [\alpha_k(\mathbf{x})n]$, i.e. $\alpha_k(\mathbf{x}) \in [\frac{i_k(\mathbf{x})}{n}, \ \frac{i_k(\mathbf{x})+1}{n})$. Using that $\sum_{k=1}^{d+1} \alpha_k(\mathbf{x}) = 1$ we obtain

$$\sum_{k=1}^{d+1} i_k(\mathbf{x}) \in [n-d, \ n]. \tag{3.10}$$

Lemma 3.4 *Let $A = \mathbf{A}_1 \dots \mathbf{A}_{d+1}$ be d-dimensional simplex on $\mathbb{R}^d$, $p \in [1\ ,\ \infty)$, $f \in L_p(A)$ be non-negative in A and $f(\mathbf{A}_i) = 0$ for every $i = 1, \dots, d+1$. Then*

$$\|f\|_{p(A)} \le c\tau_2^-(f,A)_p.$$

Proof. Let $n > d$ and $i = 1, ..., n-1$ be integer. We define

$$D^{n,i}_{\mathbf{x}} f(\mathbf{y}) \overset{\text{def}}{=} -if(\mathbf{y}) + nf(\mathbf{x}) - (n-i)f(\mathbf{y} + \frac{n}{n-i}(\mathbf{x}-\mathbf{y})). \tag{3.11}$$

We need the following inequality

$$D^{n,i}_{\mathbf{x}} f^p(\mathbf{A}_k) \le (D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k))^p_+ \quad for \ \ \mathbf{x} \in M^n_{k,i}. \tag{3.12}$$

From (3.11) and $f(\mathbf{A}_k) = 0$ it follows that

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{n} D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k) + \frac{i}{n} f(\mathbf{A}_k) + \frac{n-i}{n} f(\mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)) \\ &= \frac{1}{n} D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k) + \frac{i-1}{n} f(\mathbf{A}_k) + \frac{n-i}{n} f(\mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)) \end{aligned}$$

Hence the trivial inequality $D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k) \le (D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k))_+$ and convexity of x^p for $p \ge 1$ give

$$\begin{aligned} f^p(\mathbf{x}) &\le \Big(\frac{1}{n}(D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k))_+ + \frac{i-1}{n} f(\mathbf{A_k}) + \frac{n-i}{n} f(\mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k))\Big)^p \\ &\le \frac{1}{n}(D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k))^p_+ + \frac{i-1}{n} f^p(\mathbf{A}_k) + \frac{n-i}{n} f^p(\mathbf{A_k} + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)). \end{aligned}$$

Also from (3.11) and $f^p(\mathbf{A}_k) = 0$ it follows that

$$\begin{aligned} f^p(\mathbf{x}) &= \frac{1}{n} D^{n,i}_{\mathbf{x}} f^p(\mathbf{A}_k) + \frac{i}{n} f^p(\mathbf{A}_k) + \frac{n-i}{n} f^p(\mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)) \\ &= \frac{1}{n} D^{n,i}_{\mathbf{x}} f^p(\mathbf{A}_k) + \frac{i-1}{n} f^p(\mathbf{A}_k) + \frac{n-i}{n} f^p(\mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)). \end{aligned}$$

The last two inequalities prove (3.12). Using (3.9), (3.10) and (3.12) we derive

$$\begin{aligned} &\sum_{k=1}^{d+1} \sum_{i=1}^{n-1} \int_{M^n_{k,i}} \Big(D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k)\Big)^p_+ d\mathbf{x} \\ &\ge \sum_{k=1}^{d+1} \sum_{i=1}^{n-1} \int_{M^n_{k,i}} n f^p(\mathbf{x}) - (n-i) f^p(\mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)) d\mathbf{x} \\ &= n \left(\sum_{k=1}^{d+1} \sum_{i=1}^{n-1} \int_{M^n_{k,i}} f^p(\mathbf{x}) d\mathbf{x} - (d+1) \sum_{i=1}^{n-1} \left(\frac{n-i}{n}\right)^{d+1} \int_A f^p(\mathbf{x}) d\mathbf{x} \right) \\ &= n \left(\sum_{k=1}^{d+1} \sum_{i=1}^{n-1} i \int_{M^n_{k,i} \setminus M^n_{k,i+1}} f^p(\mathbf{x}) d\mathbf{x} - (d+1) \sum_{i=1}^{n-1} \left(\frac{i}{n}\right)^{d+1} \int_A f^p(\mathbf{x}) d\mathbf{x} \right) \\ &= n \left(\sum_{k=1}^{d+1} \int_A i_k(\mathbf{x}) f^p(\mathbf{x}) d\mathbf{x} - (d+1) n \sum_{i=1}^{n-1} \frac{1}{n} \left(\frac{i}{n}\right)^{d+1} \int_A f^p(\mathbf{x}) d\mathbf{x} \right) \\ &\ge n \left((n-d) - (d+1) n \int_0^1 t^{d+1} dt \right) \int_A f^p(\mathbf{x}) d\mathbf{x} \\ &= n \frac{n - d(d+2)}{d+2} \int_A f^p(\mathbf{x}) d\mathbf{x}. \end{aligned} \tag{3.13}$$

Here in third line we use that the map $\mathbf{x} \mapsto \mathbf{A}_k + \frac{n}{n-i}(\mathbf{x} - \mathbf{A}_k)$ is affine with a center on $\mathbf{A}_k$ and stretchs the simplex $M^n_{k,i}$ to the simplex A.

Immediately from (3.11) (the definition of $D^{n,i}_{\mathbf{x}} f(\mathbf{y})$) we have

$$D^{n,1}_{\mathbf{x}} f(\mathbf{y}) = \sum_{j=1}^{n-1} j\tilde{\Delta}^2_{\frac{\mathbf{x}-\mathbf{y}}{n-1}} f(\mathbf{y} + j\frac{\mathbf{x}-\mathbf{y}}{n-1})$$

and

$$D^{n,i}_{\mathbf{x}} f(\mathbf{y}) = iD^{n-i+1,1}_{\mathbf{x}} f(\mathbf{y}) + (n-i)D^{i,1}_{\mathbf{y}+\frac{n-i+1}{n-i}(\mathbf{x}-\mathbf{y})} f(\mathbf{y} + \frac{n}{n-i}(\mathbf{x}-\mathbf{y}))$$

Hence

$$D^{n,i}_{\mathbf{x}} f(\mathbf{y}) = i\sum_{s=1}^{n-i} s\tilde{\Delta}^2_{\frac{\mathbf{x}-\mathbf{y}}{n-i}} f(\mathbf{y} + s\frac{\mathbf{x}-\mathbf{y}}{n-i}) + (n-i)\sum_{s=n-i+1}^{n-1} (n-s)\tilde{\Delta}^2_{\frac{\mathbf{x}-\mathbf{y}}{n-i}} f(\mathbf{y} + s\frac{\mathbf{x}-\mathbf{y}}{n-i}).$$

Using the above equality, (3.12) and the definition (1.6) we obtain

$$\begin{aligned}
(3.14)\qquad & \left\{\sum_{k=1}^{d+1}\sum_{i=1}^{n-1}\int_{M^n_{k,i}} \left(D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k)\right)_+^p d\mathbf{x}\right\}^{\frac{1}{p}} \\
& \le \sum_{k=1}^{d+1}\sum_{i=1}^{n-1}\left\{\int_{M^n_{k,i}} \left(D^{n,i}_{\mathbf{x}} f(\mathbf{A}_k)\right)_+^p d\mathbf{x}\right\}^{\frac{1}{p}} \\
& \le \sum_{k=1}^{d+1}\sum_{i=1}^{n-1}\left(i\sum_{s=1}^{n-i} s\left\{\int_{M^n_{k,i}} \left(\tilde{\Delta}^2_{\frac{\mathbf{x}-\mathbf{A}_k}{n-i}} f(\mathbf{A}_k + s\frac{\mathbf{x}-\mathbf{A}_k}{n-i})\right)_+^p d\mathbf{x}\right\}^{\frac{1}{p}}\right. \\
& \left. +(n-i)\sum_{s=n-i+1}^{n-1}(n-s)\left\{\int_{M^n_{k,i}} \left(\tilde{\Delta}^2_{\frac{\mathbf{x}-\mathbf{A}_k}{n-i}} f(\mathbf{A}_k + s\frac{\mathbf{x}-\mathbf{A}_k}{n-i})\right)_+^p d\mathbf{x}\right\}^{\frac{1}{p}}\right) \\
& \le \sum_{k=1}^{d+1}\sum_{i=1}^{n-1}\left(i\sum_{s=1}^{n-i} s\left\{\int_{M^n_{k,i}} \left(\sup\{\tilde{\Delta}^2_{\mathbf{h},A} f(\mathbf{A}_k + s\frac{\mathbf{x}-\mathbf{A}_k}{n-i}) \mid \mathbf{h}\in\mathbb{R}^d\}\right)^p d\mathbf{x}\right\}^{\frac{1}{p}}\right. \\
& \left. +(n-i)\sum_{s=n-i+1}^{n-1}(n-s)\left\{\int_{M^n_{k,i}} \left(\sup\{\tilde{\Delta}^2_{\mathbf{h},A} f(\mathbf{A}_k + s\frac{\mathbf{x}-\mathbf{A}_k}{n-i}) \mid \mathbf{h}\in\mathbb{R}^d\}\right)^p d\mathbf{x}\right\}^{\frac{1}{p}}\right) \\
& = \sum_{k=1}^{d+1}\sum_{i=1}^{n-1}\left(i\sum_{s=1}^{n-i} s\left(\frac{n-i}{s}\right)^{\frac{d}{p}}\left\{\int_{M^n_{k,n-s}} \left(\sup\{\tilde{\Delta}^2_{\mathbf{h},A} f(\mathbf{x}) \mid \mathbf{h}\in\mathbb{R}^d\}\right)^p d\mathbf{x}\right\}^{\frac{1}{p}}\right. \\
& \left. +(n-i)\sum_{s=n-i+1}^{n-1}(n-s)\left(\frac{n-i}{s}\right)^{\frac{d}{p}}\left\{\int_{M^n_{k,n-s}} \left(\sup\{\tilde{\Delta}^2_{\mathbf{h},A} f(\mathbf{x}) \mid \mathbf{h}\in\mathbb{R}^d\}\right)^p d\mathbf{x}\right\}^{\frac{1}{p}}\right) \\
& \le \left[\sum_{k=1}^{d+1}\sum_{i=1}^{n-1}\left(i\sum_{s=1}^{n-i} s\left(\frac{n-i}{s}\right)^{\frac{d}{p}} + (n-i)\sum_{s=n-i+1}^{n-1}(n-s)\left(\frac{n-i}{s}\right)^{\frac{d}{p}}\right)\right] \\
& \times\left\{\int_A \left(\sup\{\tilde{\Delta}^2_{\mathbf{h},A} f(\mathbf{x}) \mid \mathbf{h}\in\mathbb{R}^d\}\right)^p d\mathbf{x}\right\}^{\frac{1}{p}} \\
& \le c_n\tau_2^-(f,A)_p.
\end{aligned}$$

The inequalites (3.13) and (3.14) with $n = (d+1)^2$ prove the lemma.

□

Let $U \subset \mathbb{R}^d$ be a polytope and let $f \in L_p(U)$ $(p \in [1,\infty)$) be bounded from below. We set

$$(3.15)\ \ C_U f(\mathbf{x}) \stackrel{\text{def}}{=} \inf\left\{\sum_{i=1}^{d+1}\alpha_i f(\mathbf{x}_i) \mid \mathbf{x} = \sum_{i=1}^{d+1}\alpha_i \mathbf{x}_i,\ \sum_{i=1}^{d+1}\alpha_i = 1,\ \alpha_i \geq 0,\ \mathbf{x}_i \in U,\ i = 1,...,d+1\right\}.$$

Immediately from (3.15) and [6] we have

(3.16) *$C_U f$ is convex on U, continuous on every open subset of U and Lipschitz function on every compact subset of the interior of U;*

(3.17) *If h is convex and is majorized by f on U then $h(\mathbf{x}) \leq C_U f(\mathbf{x})$ for all $\mathbf{x} \in U$, i.e. $C_U f$ is the biggest convex minorant of f in U.*

For $g : U \to \mathbb{R}$, let *epigraph* of g be the set $epi(g) = \{(\mathbf{x}, t) \in \mathbb{R}^{d+1} \mid \mathbf{x} \in U,\ t \geq g(\mathbf{x})\}$. We say $\mathbf{x} \in U$ is *extreme with respect to the convex function* g if $g(\mathbf{x}) < \infty$ and g is not linear on any relatively open segment containing $\mathbf{x}$. Hence $\mathbf{x}$ is extreme with respect to g if and only if $(\mathbf{x}, g(\mathbf{x}))$ is an extreme point of $epi(g)$.

Let $EP(g) \subset U$ be the set of extreme points with respect to the convex function g. Then from (3.15) and (3.17) it follows that

(3.18) *for any positive ϵ and for any $\mathbf{x} \in EP(C_U f)$ there exists $\mathbf{y} \in U$, such that $\|\mathbf{x} - \mathbf{y}\| < \epsilon$ and $|f(\mathbf{y}) - C_U f(\mathbf{x})| < \epsilon$.*

Lemma 3.5 *Let $\Pi = \Pi[\mathbf{a}; \mathbf{b}]$ and $f \in L_p(\Pi)$ ($p \in [1, \infty)$) be bounded from below. Then*

$$\|f - C_\Pi f\|_{p(\Pi)} \leq \tau_2^-(f, \Pi)_p.$$

Proof. We denote with $\mathbf{A}_1, ..., \mathbf{A}_{2^d}$ the vertices of Π. From (3.15) it follows that $\inf\{f(\mathbf{y}) \mid \mathbf{y} \in \Pi\} \leq C_\Pi f(\mathbf{x}) \leq f(\mathbf{x})$ *for every* $\mathbf{x} \in \Pi$ and hence $f - C_\Pi f \in L_p(\Pi)$.

Let ϵ be arbitrary positive. From the absolute continuity of the Lebesgue integral there is a positive $\eta < \frac{1}{2}\min\{\max\{a_i; b_i\} - \min\{a_i; b_i\} \mid i = 1, ..., d\}$, such that

$$(3.19)\qquad \|f - C_\Pi f\|_{p(\Pi \setminus \Pi(\eta))} \leq \epsilon,$$

where $\Pi(\eta) = \Pi[\mathbf{c}; \mathbf{h}]$, $c_i \stackrel{\text{def}}{=} \min\{a_i; b_i\} + \eta$ and $h_i \stackrel{\text{def}}{=} \max\{a_i; b_i\} - \eta$ for every $i = 1, ..., d$.

The set $\Pi(\eta)$ is compact subset of the interior of Π and then (3.16) gives that there exists a positive constant L such that

(3.20) $\quad |C_\Pi f(\mathbf{x}) - C_\Pi f(\mathbf{y})| \leq L\|\mathbf{x} - \mathbf{y}\|$ *for any two points* $\mathbf{x},\ \mathbf{y} \in \Pi(\eta)$.

For $i = 1, ..., d$ we set $n_i \stackrel{\text{def}}{=} \left[\epsilon\frac{|h_i - c_i|}{3L}\right] + 1$ and let $\mathbb{Z}(\epsilon) \stackrel{\text{def}}{=} \left\{\mathbf{j} \in \mathbb{Z}^d \mid j_i \in [0, n_i],\ i = 1, ..., d\right\}$. For every $\mathbf{j} \in \mathbb{Z}(\epsilon)$ we set $\mathbf{z_j} \stackrel{\text{def}}{=} \left(c_1 + j_1\frac{(h_1 - c_1)}{n_1}, ..., c_d + j_d\frac{(h_d - c_d)}{n_d}\right) \in \Pi(\eta)$.

As a corollary from the theorem of J.-C. Aggeri (Krein-Milman's type theorem for convex functions (see [1])) we derive that for every $\mathbf{x} \in \Pi(\eta)$ and every $\delta > 0$ there exist points $\mathbf{y}_i(\mathbf{x}, \delta) \in EP(C_\Pi f)$ $i = 1, ..., d+1$ with $\mathbf{x} = \sum_{i=1}^{d+1} \alpha_i \mathbf{y}_i(\mathbf{x}, \delta)$,
$\sum_{i=1}^{d+1} \alpha_i = 1,\ \alpha_i \geq 0$ for which $C_\Pi f(\mathbf{x}) \geq \sum_{i=1}^{d+1} \alpha_i C_\Pi f(\mathbf{y}_i(\mathbf{x}, \delta)) - \delta$.
Let $EP(\epsilon) \stackrel{\text{def}}{=} \left\{ \mathbf{y}_i(\mathbf{z}_\mathbf{j}, \frac{\epsilon}{3}) \mid i = 1, ..., d+1,\ \mathbf{j} \in \mathbb{Z}(\epsilon) \right\}$.
This is a $m-$ points set where $m \leq (d+1) \prod_{i=1}^{d} (n_i + 1)$.

We define

$$s_1(\mathbf{x}) \stackrel{\text{def}}{=} \min \left\{ \sum_{i=1}^{d+1} \alpha_i C_\Pi f(\mathbf{a}_i) \mid \mathbf{x} = \sum_{i=1}^{d+1} \alpha_i \mathbf{a}_i,\ \sum_{i=1}^{d+1} \alpha_i = 1,\ \alpha_i \geq 0,\ \mathbf{a}_i \in EP(\epsilon) \right\}.$$

This is a first degree convex interpolation spline for $C_\Pi f$ with knots in $EP(\epsilon)$.

For every $\mathbf{x} \in \Pi(\eta)$ we have that there exist set of points $\left\{ \mathbf{z}_{\mathbf{j}(\mathbf{x},i)} \right\}_{i=1}^{d+1}$, such that $\mathbf{x} = \sum_{i=1}^{d+1} \alpha_i \mathbf{z}_{\mathbf{j}(\mathbf{x},i)},\ \sum_{i=1}^{d+1} \alpha_i = 1,\ \alpha_i \geq 0,\ \mathbf{j}(\mathbf{x}, i) \in \mathbb{Z}(\epsilon)$ and $|\mathbf{z}_{\mathbf{j}(\mathbf{x},i)} - \mathbf{x}| \leq \frac{\epsilon}{3L}$ for $i = 1, ..., d+1$
For this points (3.20) gives

$$|C_\Pi f(\mathbf{x}) - C_\Pi f(\mathbf{z}_{\mathbf{j}(\mathbf{x},i)})| \leq \frac{\epsilon}{3} \ \ \forall i = 1, ..., d+1$$

Using the definitions of $s_1(\mathbf{x})$, $EP(\epsilon)$, the points $\left\{ \mathbf{z}_{\mathbf{j}(\mathbf{x},i)} \right\}_{i=1}^{d+1}$ and the last inequality we obtain

$$\begin{aligned} 0 \leq s_1(\mathbf{x}) - C_\Pi f(\mathbf{x}) &\leq \sum_{i=1}^{d+1} \alpha_i \sum_{k=1}^{d+1} \alpha_{i,k} C_\Pi f \left(\mathbf{y}_k \left(\mathbf{z}_{\mathbf{j}(\mathbf{x},i)}, \frac{\epsilon}{3} \right) \right) - C_\Pi f(\mathbf{x}) \\ &\leq \sum_{i=1}^{d+1} \alpha_i \left(C_\Pi f(\mathbf{z}_{\mathbf{j}(\mathbf{x},i)}) + \frac{\epsilon}{3} \right) - C_\Pi f(\mathbf{x}) \\ &\leq \sum_{i=1}^{d+1} \alpha_i \left((C_\Pi f(\mathbf{x}) + \frac{\epsilon}{3}) + \frac{\epsilon}{3} \right) - C_\Pi f(\mathbf{x}) \\ &\leq \epsilon \frac{2}{3} \end{aligned}$$

Using s_1 and (3.18) we can find a first degree interpolation spline $s(\mathbf{x})$ with knots $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\} \in \Pi$ such that $C_\Pi f(\mathbf{x}) \leq s(\mathbf{x}) \leq C_\Pi f(\mathbf{x}) + \epsilon$ for $\mathbf{x} \in \Pi(\eta)$ and $f(\mathbf{y}_i) = s(\mathbf{y}_i)$, $i = 1, \ldots, m$.

Suppose $\Pi(\eta) \subset \cup_{i=1}^{k} D_i$ where $D_i = \mathbf{y}_{i_1} \ldots \mathbf{y}_{i_{d+1}}$ are d-dimensional simplecies with $\{\mathbf{y}_{i_1}, \ldots, \mathbf{y}_{i_{d+1}}\} \subset \{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ and the restrictions of $s(\mathbf{x})$ on D_i are affine functions.
Then from (3.19)

$$\begin{aligned} \|f - C_\Pi f\|_{p(\Pi)}^p &= \|f - C_\Pi f\|_{p(\Pi(\eta))}^p + \|f - C_\Pi f\|_{p(\Pi \setminus \Pi(\eta))}^p \\ &\leq \sum_{i=1}^{k} \|f - C_\Pi f\|_{p(D_i \cap \Pi(\eta))}^p + \epsilon^p. \end{aligned} \tag{3.21}$$

Using the definitions of $C_\Pi f$ and s, trivial equality $(f - s) = (f - s)_+ - (s - f)_+$, applying Lemma 3.4 to $(f - s)_+$ in D_i and properties (3.1), (3.2) and (3.4), we get

$$\|f - C_\Pi f\|_{p(D_i \cap \Pi(\eta))} \leq \|f - s\|_{p(D_i \cap \Pi(\eta))} + \|s - C_\Pi f\|_{p(D_i \cap \Pi(\eta))} \tag{3.22}$$

$$
\begin{aligned}
&\leq \|(f-s)_+\|_{p(D_i\cap\Pi(\eta))} + \|(s-f)_+\|_{p(D_i\cap\Pi(\eta))} + \epsilon\mu(D_i\cap\Pi(\eta))^{\frac{1}{p}} \\
&\leq \|(f-s)_+\|_{p(D_i)} + \|(s-f)_+\|_{p(D_i\cap\Pi(\eta))} + \epsilon\mu(D_i\cap\Pi(\eta))^{\frac{1}{p}} \\
&\leq c\tau_2^-((f-s)_+, D_i)_p + \|s - C_\Pi f\|_{p(D_i\cap\Pi(\eta))} + \epsilon\mu(D_i\cap\Pi(\eta))^{\frac{1}{p}} \\
&\leq c\tau_2^-((f-s), D_i)_p + 2.\epsilon\mu(D_i\cap\Pi(\eta))^{\frac{1}{p}} \\
&\leq c\tau_2^-((f-s), D_i)_p + 2.\epsilon\mu(D_i)^{\frac{1}{p}} \\
&\leq c\tau_2^-(f, D_i)_p + 2\epsilon\mu(D_i)^{\frac{1}{p}}.
\end{aligned}
$$

The inequalities (3.21) and (3.22) give

$$
\begin{aligned}
\|f - C_\Pi f\|_{p(\Pi)} &\leq c\left\{\sum_{i=1}^{k}(\tau_2^-(f, D_i)_p + 2\epsilon\mu(D_i)^{\frac{1}{p}})^p\right\}^{\frac{1}{p}} + \epsilon \\
&\leq c\left\{\sum_{i=1}^{k}\tau_2^-(f, D_i)_p^p\right\}^{\frac{1}{p}} + 2\epsilon(\sum_{i=1}^{k}\mu(D_i))^{\frac{1}{p}} + \epsilon \\
&\leq c\tau_2^-(f,\Pi)_p + (2\mu(\Pi)+1)\epsilon \\
&\leq c(\tau_2^-(f,\Pi)_p + \epsilon).
\end{aligned}
$$

Lemma 3.5 is proved. □

Lemma 3.6 *Under the assumtion of Lemma 3.5 we have*

$$
E^-(f, H_1)_{p(\Pi)} \sim E(f, H_1)_{p(\Pi)} + \tau_2^-(f,\Pi)_p.
$$

Proof. The inequality $C_\Pi f \leq f$ implies

$$
E^-(f, H_1)_{p(\Pi)} \leq E^-(C_\Pi f, H_1)_{p(\Pi)} + \|f - C_\Pi f\|_{p(\Pi)}.
$$

Lemma 3.3 applied to $C_\Pi f$ gives

$$
E^-(C_\Pi f, H_1)_{p(\Pi)} \leq cE(C_\Pi f, H_1)_{p(\Pi)}.
$$

Combining the above two inequalities and

$$
E(C_\Pi f, H_1)_{p(\Pi)} \leq E(f, H_1)_{p(\Pi)} + \|f - C_\Pi f\|_{p(\Pi)}
$$

we prove the direct inequality in view of Lemma 3.5. In order to get the other direction of the equivalence in Lemma 3.6 we estimate both terms of its right-hand side by $E^-(f, H_1)_{p(\Pi)}$. Obviously $E(f, H_1)_{p(\Pi)} \leq E^-(f, H_1)_{p(\Pi)}$. Let $Q \in H_1$ be such that $f \geq Q$ and $E^-(f, H_1)_{p(\Pi)} = \|f - Q\|_{p(\Omega)}$. From the property (3.1) we have

$$
\tau_2^-(f,\Pi)_p = \tau_2^-(f - Q,\Pi)_p \leq 2\|f - Q\|_{p(\Pi)} = 2E^-(f, H_1)_{p(\Pi)}.
$$

Hence

$$E(f,H_1)_{p(\Pi)} + \tau_2^-(f,\Pi)_p \le 3E^-(f,H_1)_{p(\Pi)}.$$

□

Proof of Theorem 1.3(II). Using that $E(f,H_{r-1})_{p(\Pi)} \sim \omega_r(f,\Pi)_p$ (see[4]), as a corollary from the last lemma we obtain Theorem 1.3(II). Here the result for $p=\infty$ is trivial. □

Proof of Theorem 1.7. From the definition we have that the almost midconvex function is bounded from above. Then Theorem 1.7 follows from Lemma 3.5. □

4 Main results.

Proof of Theorem 1.5.

Here we use the ideas from [1]. Utilizing Theorem 1.3(I) and Theorem 1.2 we obtain a characterization of $K^-(f,\Psi(t),L_p,W_p^1,W_p^{l_1})$. We demonstrate the proof in the more complicated case- $r=2$.

Using $K_2^-(f,\rho t) \le max\{1,\rho^{l_2}\}K_2^-(f,t)$ for $\rho>0$, Theorem 1.2, Theorem 1.3(II), Theorem 1.4 (with $r=2$, $\pi=\frac{1}{2}B(t,\mathbf{x})$, $\Pi=U(t,\mathbf{x})$ and $R=2$), (1.8) and (1.9) we have

$$\begin{aligned}
(4.1)\qquad K_2^-(f,t)_p &\sim K_2^-(f,\rho t)_p \\
&\sim \|\Psi(t,\cdot)^{-\frac{1}{p}}E^-(f,H_{r-1})_{p(U(t,\cdot))}\|_{p(\Omega)} \\
&\sim \|\Psi(t,\cdot)^{-\frac{1}{p}}\{\omega_2(f,U(\rho t,\cdot))_p + \tau_2^-(f,U(\rho t,\cdot))_p\}\|_{p(\Omega)} \\
&\sim \|\Psi(t,\cdot)^{-\frac{1}{p}}\{\tau_2(f,\frac{1}{2}B(\rho t,\cdot))_{p,p(U(\rho t,\cdot))} + \tau_2^-(f,U(\rho t,\cdot))_p\}\|_{p(\Omega)} \\
&\sim \left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\rho t,\cdot)}\Psi(\rho t,\cdot)^{-1}\int_{\frac{1}{2}B(\rho t,\cdot)}|\Delta^2_{\mathbf{v},U(\rho t,\cdot)}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)} \\
&+\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\rho t,\cdot)}\left[\sup\{\tilde{\Delta}^2_{\mathbf{h},U(\rho t,\cdot)}f(\mathbf{y}) \mid \mathbf{h}\in\mathbb{R}^d\}\right]^p d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}.
\end{aligned}$$

From (2.4) with $d=1$ and the definitions of $\Psi(t,\mathbf{x})$ and $U(t,\mathbf{x})$ (see also [2], Lemma 3) we have

$$(4.2)\qquad \Psi(\frac{1}{6}t,\mathbf{x}) \le \Psi(t,\mathbf{y}) \quad for \quad every \quad \mathbf{y}\in U(\frac{1}{6}t,\mathbf{x}) \quad and \quad \mathbf{x}\in\Omega \quad and$$

$$(4.3)\qquad \Psi(t,\mathbf{y}) \le \Psi(4t,\mathbf{x}) \quad for \quad every \quad \mathbf{y}\in U(t,\mathbf{x}) \quad and \quad \mathbf{x}\in\Omega.$$

Then using (2.3), (2.4), (4.2) and Lemma 2.1 we get

$$\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\frac{1}{6}t,\cdot)}\Psi(\frac{1}{6}t,\cdot)^{-1}\int_{\frac{1}{2}B(\frac{1}{6}t,\cdot)}|\Delta^2_{\mathbf{v},U(\frac{1}{6}t,\cdot)}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}$$
$$\leq c\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\frac{1}{6}t,\cdot)}\Psi(t,\mathbf{y})^{-1}\int_{\frac{1}{2}B(t,\mathbf{y})}|\Delta^2_{\mathbf{v},U(t,\mathbf{y})}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}$$
$$\sim\left\|\Psi(\frac{1}{6}t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\frac{1}{6}t,\cdot)}\Psi(t,\mathbf{y})^{-1}\int_{\frac{1}{2}B(t,\mathbf{y})}|\Delta^2_{\mathbf{v},U(t,\mathbf{y})}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}$$
$$\sim \tau_2(f,\Psi(t))_{p,p(\Omega)}.$$

Using the same arguments we have that

$$\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\frac{1}{6}t,\cdot)}\sup\{\tilde{\Delta}^2_{\mathbf{h},U(\frac{1}{6}t,\cdot)}f(\mathbf{y})\ \mid\ \mathbf{h}\in\mathbb{R}^d\}^p d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}\leq c\tau_2^-(f,\Psi(t))_{p(\Omega)}.$$

Then from (4.1) with $\rho=\frac{1}{6}$ we get the inequality

$$K^-(f,\Psi(t),L_p,W_p^2,W_p^{l_2})\leq c\{\tau_2^-(f,\Psi(t))_{p(\Omega)}+\tau_2(f,\Psi(t))_{p,p(\Omega)}\}.$$

The proof of the opposite inequality is the same. Using Lemma 2.1, (4.3) and (2.4) we get

$$\tau_2(f,\Psi(t))_{p,p(\Omega)}$$
$$\sim\left\|\Psi(\frac{1}{2}t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\frac{1}{2}t,\cdot)}\Psi(t,\mathbf{y})^{-1}\int_{\frac{1}{2}B(t,\mathbf{y})}|\Delta^2_{\mathbf{v},U(t,\mathbf{y})}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}$$
$$\leq c\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(\frac{1}{6}t,\cdot)}\Psi(t,\mathbf{y})^{-1}\int_{\frac{1}{2}B(t,\mathbf{y})}|\Delta^2_{\mathbf{v},U(t,\mathbf{y})}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}$$
$$\leq c\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(4t,\cdot)}\Psi(4t,\cdot)^{-1}\int_{\frac{1}{2}B(4t,\cdot)}|\Delta^2_{\mathbf{v},U(4t,\cdot)}f(\mathbf{y})|^p d\mathbf{v}d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}.$$

In the same way we have that

$$\tau_2^-(f,\Psi(t))_{p(\Omega)}\leq c\left\|\Psi(t,\cdot)^{-\frac{1}{p}}\left\{\int_{U(4t,\cdot)}\sup\{\tilde{\Delta}^2_{\mathbf{h},U(4t,\cdot)}f(\mathbf{y})\ \mid\ \mathbf{h}\in\mathbb{R}^d\}^p d\mathbf{y}\right\}^{\frac{1}{p}}\right\|_{p(\Omega)}.$$

Then from (4.1) with $\rho=4$ we get the inequality

$$\tau_2^-(f,\Psi(t))_{p(\Omega)}+\tau_2(f,\Psi(t))_{p,p(\Omega)}\leq cK^-(f,\Psi(t),L_p,W_p^2,W_p^{l_2}).$$

□

The following result for the unconstrained K-functional is valid (see [4] Theorem 1.3).

Lemma 4.1

$$c\tau_r(f,\Psi(t))_{p,p(\Omega)} \le K\left(f,\Psi(t),L_p,W_p^r\right) \le c\tau_r(f,\Psi(t))_{p,p(\Omega)}$$

Proof of Theorem 1.6. Applying Theorem 1.5 together with Theorem 1.1 and Lemma 4.1 we obtain Theorem 1.6. □

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