Weighted Approximation by the Goodman–Sharma Operators

K. G. Ivanov, P. E. Parvanov*

April 25, 2010

Abstract

The uniform weighted approximation errors of the Goodman–Sharma operators are characterized for functions from $C(w)[0,1]$ with weight of the form $x^{\gamma_0}(1-x)^{\gamma_1}$ for $\gamma_0, \gamma_1 \in [-1,0]$. Direct and strong converse theorems are proved in terms of the weighted K-functional. The results extends those in [6] from the unweighted case ($\gamma_0 = \gamma_1 = 0$) to weights with negative powers.

AMS classification: 41A36, 41A10, 41A25, 41A27, 41A17.

Keywords: Bernstein-type operator, Direct theorem, Strong converse theorem, K-functional.

1 Introduction

The Bernstein-type operators discussed in this paper are given for natural n by

$$\begin{aligned} U_n f(x) &= \sum_{k=0}^{n} u_{n,k}(f) P_{n,k}(x) \\ &= f(0)P_{n,0}(x) + f(1)P_{n,n}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(y)f(y)dy, \end{aligned}$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and f is a Lebesgue integrable in $(0,1)$ function with finite limits at 0 and at 1. They were introduced by T.N.T. Goodman and A. Sharma in [4] and [5]. These operators can also be considered as a limit case of the family of Bernstein-type operators investigated by H. Berens and Y. Xu in [1].

*Partially supported by grant Nr103/2007 of the National Science Fund of the Sofia University

Denote the weight function by

$$w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1-x)^{\gamma_1} \text{ for } x \in (0,1) \tag{1.1}$$

and real γ_0, γ_1. Our main results will concern the values of the powers γ_0, γ_1 in the range $[-1, 0]$. By $\varphi(x) = x(1-x)$ we denote the weight which is naturally connected with the second derivatives of both Bernstein and Goodman–Sharma operators. The first derivative operator is given by $D = \frac{d}{dx}$, thus $Dg(x) = g'(x)$ and $D^2g(x) = g''(x)$.

By $C[0,1]$, as usual, we denote the space of all continuous functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|$. Let $L_\infty[0,1]$ denote the Lebesgue measurable and essentially bounded in $[0,1]$ functions. For a weight function w we set $C(w)[0,1] = \{f \in C[0,1] : wf \in L_\infty[0,1]\}$ and

$$W^2(w\varphi)[0,1] = \left\{g, g' \in AC_{loc}(0,1) : w\varphi D^2 g \in L_\infty[0,1]\right\},$$

where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$.

Set $C_0(w)[0,1] = \{f \in C(w)[0,1] : f(0) = f(1) = 0\}$. Similarly, by $W_0^2(w\varphi)[0,1]$ we denote the subspace of $W^2(w\varphi)[0,1]$ of functions g satisfying the additional boundary conditions

$$\lim_{x\to 0+0} \varphi(x)D^2g(x) = \lim_{x\to 1-0} \varphi(x)D^2g(x) = 0.$$

Note that the boundary conditions both for $C_0(w)$ and $W_0^2(w\varphi)$ do not depend on the weight w. These conditions are essential when the weight w does not go to ∞ at least at one of the end-points of $[0,1]$, while for $\gamma_0, \gamma_1 < 0$ we have $C_0(w)[0,1] = C(w)[0,1]$ and $W_0^2(w\varphi)[0,1] = W^2(w\varphi)[0,1]$. The above defined spaces are naturally embedded

$$W_0^2(w\varphi)[0,1] \subset W^2(w\varphi)[0,1] \subset C(w)[0,1] + \pi_1 \subset C[0,1], \tag{1.2}$$

where π_1 is the set of all algebraical polynomials of degree 1. Note that $W_0^2(w\varphi)[0,1] \not\subset C_0[0,1] = C_0(1)[0,1]$ because $\pi_1 \subset W_0^2(w\varphi)[0,1]$.

In this paper we investigate the rate of weighted approximation by U_n for functions in $C_0(w)[0,1] + \pi_1$. The weighted approximation error will be compared with the K-functional between the weighted spaces $C(w)[0,1]$ and $W^2(w\varphi)[0,1]$, which for every $f \in C(w)[0,1] + \pi_1$ and $t > 0$ is defined by

$$K_w(f,t) = \inf\left\{\|w(f-g)\| + t\|w\varphi D^2 g\| : g \in W^2(w\varphi)[0,1]\right\}. \tag{1.3}$$

Goodman–Sharma operators combine good properties both of Bernstein operators and of their Durrmeyer modification. Thus, Goodman–Sharma operators U_n like Bernstein operators preserve linear functions and are suitable for uniform approximation. On the other hand, U_n like Bernstein-Durrmeyer operators commute among themselves ($U_nU_mf = U_mU_nf$) and with the differential operator φD^2 (see Lemma 2.2). The last property simplifies essentially the proof of the strong converse theorem for Goodman–Sharma operators.

Our main result is the following theorem, consisting of a direct inequality (1.4) and a strong converse inequality of type A (1.5) in the terminology of [2]. It is a generalization of the result in [6], which treats the case $w = 1$.

Theorem 1.1. *Let $w = w(\gamma_0, \gamma_1)$ be given by (1.1) with $\gamma_0, \gamma_1 \in [-1, 0]$. There exists an absolute constant M such that for every $f \in C(w)[0,1] + \pi_1$ and every $n \in \mathbb{N}$ we have*

$$\|w(f - U_n f)\| \le 2K_w\left(f, \frac{1}{2n}\right), \tag{1.4}$$

$$K_w\left(f, \frac{1}{2n}\right) \le \left(\frac{162 + 9\sqrt{2}}{28} + \frac{M}{n}\right)\|w(f - U_n f)\|. \tag{1.5}$$

Note that both sides of (1.4) (and (1.5)) do not change if f is replaced by $f - q$ for any $q \in \pi_1$. Hence, it is enough to prove Theorem 1.1 for functions $f \in C(w)[0,1]$. Inequality (1.4) is contained in the following direct theorem, because $w(\gamma_0, \gamma_1)^{-1}$ is concave for $\gamma_0, \gamma_1 \in [-1, 0]$.

Theorem 1.2. (Direct theorem) *Let w^{-1} be concave. Then for every $f \in C(w)[0,1]$ and $n \in \mathbb{N}$ we have*

$$\|w(U_n f - f)\| \le 2K_w\left(f, \frac{1}{2n}\right). \tag{1.6}$$

Inequality (1.5) follows from the following inverse theorem, because $\kappa(n) = 1 + O(n^{-1})$. Note that for $n = 1$ we have $D^2 U_1 f = 0$ and, thus, inequality (1.5) is trivially satisfied with a constant 1.

Theorem 1.3. (Strong inverse theorem of type A) *Let $w = w(\gamma_0, \gamma_1)$ be given by (1.1) with $\gamma_0, \gamma_1 \in [-1, 0]$. For every $f \in C(w)[0,1]$ and for every $n \in \mathbb{N}$, $n \ge 2$, we have*

$$K_w\left(f, \frac{1}{2n}\right) \le \frac{40 - (9\sqrt{2} - 5)\kappa(n)}{8 - 2\sqrt{2}\kappa(n)}\|w(f - U_n f)\|,$$

where $\kappa(n) = \dfrac{16n^4 + 16n^3 - 1}{16n^4 - 8n^2 + 1}$.

Remark 1.1. Although Theorem 1.1 is proved for every $f \in C(w)[0,1]$, it does not imply for such f's that $\|w(f - U_n f)\| \to 0$ or $K_w\left(f, (2n)^{-1}\right) \to 0$ when $n \to \infty$. Of course, the convergence to 0 folds for every $f \in C(w)[0,1]$ in the case $\gamma_0 = \gamma_1 = 0$. But when $\gamma_0 < 0$ or $\gamma_1 < 0$ we have to impose additional restrictions on f in the respective end-points for such convergence. These restrictions are $\lim_{x\to 0+} x^{\gamma_0} f(x) = 0$ for $-1 < \gamma_0 < 0$, the existence of $\lim_{x\to 0+} x^{-1} f(x)$ for $\gamma_0 = -1$ and similarly for γ_1. These effects are studied in [3], which also contains a characterization of the K-functional (1.3) in terms of moduli of smoothness.

The paper is organized as follows. Section 2 contains auxiliary results about Goodman–Sharma operators. Theorem 1.2 is proved in Section 3, while Theorem 1.3 is proved in Section 4.

2 Auxilary results

Many properties of the operator U_n, $n \in \mathbb{N}$ are proved in [4], [5] and [6]. We recall that U_n preserves the linear functions and $U_n f$ interpolates f at 0 and at 1. We shall also use

$$U_n \text{ is a linear, positive operator;} \tag{2.1}$$

$$U_n f \leq f \text{ for every concave continuous function } f. \tag{2.2}$$

The following two lemmas are respectively Lemma 4.1 and Lemma 4.2 in [6].

Lemma 2.1. *For every $f \in C[0,1]$ and $k \in \mathbb{N}$ we have*

$$U_k f(x) - U_{k+1} f(x) = \frac{1}{k(k+1)} \varphi(x) D^2 U_{k+1} f(x).$$

Lemma 2.2. *For every $g \in W_0^2(\varphi)[0,1]$ and $n \in \mathbb{N}$ we have*

$$\varphi(x) D^2 U_n g(x) = U_n(\varphi D^2 g)(x),$$

i.e. U_n commutes with the operator φD^2 on $W_0^2(\varphi)[0,1]$.

We also need the bounded weighted norm property of U_n and $\varphi D^2 U_n$.

Lemma 2.3. *Let w^{-1} be concave. Then for every $f \in C(w)[0,1]$ and $n \in \mathbb{N}$ we have $\|wU_n f\| \leq \|wf\|$, i.e. U_n has norm 1 in $C(w)[0,1]$.*

Proof. From (2.1) and $w \geq 0$ we get

$$|U_n f(x)| = |U_n((wf)w^{-1})(x)| \leq U_n(\|wf\| w^{-1})(x) = \|wf\| U_n(w^{-1})(x).$$

From the concavity of w^{-1} and (2.2) we get $U_n\left(w^{-1}\right) \leq w^{-1}$, which proves the lemma. □

Lemma 2.4. *Let w^{-1} be concave. Then for every $g \in W^2(w\varphi)[0,1]$ and $n \in \mathbb{N}$ we have*

$$\|w\varphi D^2 U_n g\| \leq \|w\varphi D^2 g\|.$$

Proof. From the proof of Lemma 4.2 in [6] for every $g \in W^2(w\varphi)[0,1]$ we have

$$\varphi(x) D^2 U_n g(x) = \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1) P_{n-2,k-1}(y) \varphi(y) D^2 g(y)\, dy.$$

Applying the above representation and the inequality $U_n\left(w^{-1}\right) \leq w^{-1}$ for the concave function w^{-1} as in the previous lemma we obtain

$$\begin{aligned} &|w(x)\varphi(x)D^2 U_n g(x)| \\ &\qquad \leq \|w\varphi D^2 g\|\, w(x) \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1) P_{n-2,k-1}(y) \frac{1}{w(y)}\, dy \\ &\qquad\qquad \leq \|w\varphi D^2 g\|\, w(x) U_n\left(w^{-1}\right)(x) \leq \|w\varphi D^2 g\|, \end{aligned}$$

which proves the lemma. □

3 Proof of the direct theorem

The next lemma is a weighted Jackson-type inequality for the Goodman–Sharma operators.

Lemma 3.1. *Let w^{-1} be concave. Then for every $g \in W^2(w\varphi)[0,1]$ and $n \in \mathbb{N}$ we have*

$$\|w(U_n g - g)\| \le \frac{1}{n}\|w\varphi D^2 g\|.$$

Proof. From Theorem 3.2 in [6] and (1.2) for a function $g \in W^2(w\varphi)[0,1]$ we get $\|U_n g - g\| \to 0$ for $n \to \infty$. Then from Lemma 2.1 we have

$$U_n g(x) - g(x) = \sum_{k=n}^{\infty}(U_k g(x) - U_{k+1} g(x)) = \sum_{k=n}^{\infty} \frac{\varphi(x) D^2 U_{k+1} g(x)}{k(k+1)}. \tag{3.1}$$

From (3.1) and Lemma 2.4 we get

$$\begin{aligned}\|w(U_n g - g)\| &= \left\| w \sum_{k=n}^{\infty} \frac{\varphi D^2 U_{k+1} g}{k(k+1)} \right\| \\ &\le \sum_{k=n}^{\infty} \frac{\|w\varphi D^2 U_{k+1} g\|}{k(k+1)} \le \sum_{k=n}^{\infty} \frac{1}{k(k+1)}\|w\varphi D^2 g\| = \frac{1}{n}\|w\varphi D^2 g\|.\end{aligned}$$

□

Proof of Theorem 1.2. Let g be an arbitrary function from $W^2(w\varphi)[0,1]$. Then

$$\|w(U_n f - f)\| \le \|w(U_n f - U_n g)\| + \|w(U_n g - g)\| + \|w(g - f)\|.$$

From Lemma 2.3 and Lemma 3.1 we get

$$\|w(U_n f - f)\| \le 2\|w(f-g)\| + \frac{1}{n}\|w\varphi D^2 g\| = 2\left(\|w(f-g)\| + \frac{1}{2n}\|w\varphi D^2 g\|\right).$$

Taking an infimum on $g \in W^2(w\varphi)$ in the above inequality we prove the theorem. □

4 Proof of the inverse theorem

In the proof of the inverse theorem we use the following two lemmas. The first is a strong Voronovskaya-type estimate.

Lemma 4.1. *Let w^{-1} be concave. Then for every $g \in W_0^2(w\varphi)[0,1]$ such that $\varphi D^2 g \in W^2(w\varphi)[0,1]$ and for every $n \in \mathbb{N}$ we have*

$$\left\| w\left(U_n g - g - \frac{1}{n}\varphi D^2\left(\frac{g + U_n g}{2}\right)\right)\right\| \le \frac{16n^4 + 16n^3 - 1}{4n^2(16n^4 - 8n^2 + 1)}\|w\varphi D^2(\varphi D^2 g)\|.$$

Proof. From (3.1) and Lemma 2.2 we derive the representation

$$
\begin{aligned}
&U_n g - g - \frac{1}{n}\varphi D^2\left(\frac{g+U_n g}{2}\right)\\
&\quad = \sum_{k=n+1}^{\infty}\frac{U_k(\varphi D^2 g)}{k(k-1)} - \frac{1}{2n}\varphi D^2 g - \frac{1}{2n}\varphi D^2(U_n g)\\
&\quad = \sum_{k=2n+1}^{\infty}\frac{U_k(\varphi D^2 g)-\varphi D^2 g}{k(k-1)} + \sum_{k=n+1}^{2n}\frac{U_k(\varphi D^2 g)-U_n(\varphi D^2 g)}{k(k-1)}\\
&\quad = \sum_{k=2n+1}^{\infty}\frac{U_k(\varphi D^2 g)-\varphi D^2 g}{k(k-1)} + \sum_{k=n+1}^{2n}\sum_{s=n}^{k-1}\frac{U_{s+1}(\varphi D^2 g)-U_s(\varphi D^2 g)}{k(k-1)},
\end{aligned}
$$

with the series convergent in $C(w)[0,1]$. From this representation, Lemma 3.1, Lemma 2.1 and Lemma 2.4 we get

$$
\begin{aligned}
&\|w\left(U_n g - g - \frac{1}{n}\varphi D^2\left(\frac{g+U_n g}{2}\right)\right)\|\\
&\leq \sum_{k=2n+1}^{\infty}\frac{\|w(U_k(\varphi D^2 g)-\varphi D^2 g)\|}{k(k-1)} + \sum_{k=n+1}^{2n}\sum_{s=n}^{k-1}\frac{\|w(U_{s+1}(\varphi D^2 g)-U_s(\varphi D^2 g))\|}{k(k-1)}\\
&\leq \sum_{k=2n+1}^{\infty}\frac{\|w\varphi D^2(\varphi D^2 g)\|}{k^2(k-1)} + \sum_{k=n+1}^{2n}\sum_{s=n}^{k-1}\frac{\|w\varphi D^2 U_{s+1}(\varphi D^2 g)\|}{k(k-1)s(s+1)}\\
&\leq A_n\,\|w\varphi D^2(\varphi D^2 g)\| \qquad (4.1)
\end{aligned}
$$

with

$$A_n = \sum_{k=2n+1}^{\infty}\frac{1}{k^2(k-1)} + \sum_{k=n+1}^{2n}\frac{1}{k(k-1)}\sum_{s=n}^{k-1}\frac{1}{s(s+1)}.$$

Changing the order of summation in the double sum above and using

$$
\begin{aligned}
\sum_{s=n}^{2n-1}\frac{1}{s(s+1)}\sum_{k=s+1}^{2n}\frac{1}{k(k-1)} &= \sum_{s=n}^{2n-1}\frac{1}{s(s+1)}\left(\frac{1}{s}-\frac{1}{2n}\right)\\
&= \sum_{s=n}^{2n-1}\frac{1}{s^2(s+1)} - \frac{1}{4n^2}
\end{aligned}
$$

we get

$$
\begin{aligned}
A_n &= \sum_{k=2n+1}^{\infty}\frac{1}{k^2(k-1)} + \sum_{s=n}^{2n-1}\frac{1}{s^2(s+1)} - \frac{1}{4n^2}\\
&< \frac{2n+2}{2n+1}\sum_{k=2n+1}^{\infty}\frac{1}{(k-1)k(k+1)} + \frac{2n-2}{2n-1}\sum_{s=n}^{2n-1}\frac{1}{(s-1)s(s+1)} - \frac{1}{4n^2}
\end{aligned}
$$

$$= \frac{n+1}{2n+1} \sum_{k=2n+1}^{\infty} \left(\frac{1}{k+1} - \frac{2}{k} + \frac{1}{k-1} \right)$$

$$+ \frac{n-1}{2n-1} \sum_{s=n}^{2n-1} \left(\frac{1}{s+1} - \frac{2}{s} + \frac{1}{s-1} \right) - \frac{1}{4n^2}$$

$$= \frac{16n^4 + 16n^3 - 1}{4n^2(16n^4 - 8n^2 + 1)},$$

which in view of (4.1) proves the lemma. □

The next lemma is a weighted Bernstein-type inequality for the Goodman–Sharma operators.

Lemma 4.2. *Let w be given by (1.1) with $\gamma_0, \gamma_1 \in [-1, 0]$. For every $F \in C_0(w)[0,1]$ and for every $n \in \mathbb{N}$ we have*

$$n^{-1} \| w\varphi D^2(U_n^2 F) \| \leq \sqrt{2} \| wF \|.$$

Proof. Applying Lemma 2.2 for function $g = U_n F \in W_0^2(\varphi)[0,1]$, using the formulas

$$n^{-1}(n-1)P_{n-2,k-1}(y)\varphi(y) = \varphi(kn^{-1})P_{n,k}(y), \tag{4.2}$$

$$DP_{n,k}(y) = P_{n,k}(y)(k - ny)\varphi(y)^{-1} \tag{4.3}$$

and integration by parts we get

$$\begin{aligned}
n^{-1} \left| w(x)\varphi(x)D^2 U_n(U_n F)(x) \right| &= n^{-1} \left| w(x) U_n(\varphi D^2 U_n F)(x) \right| \\
&= n^{-1} w(x) \left| \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1) P_{n-2,k-1}(y)\varphi(y) D^2 U_n F(y) dy \right| \\
&= w(x) \left| \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 \varphi\left(\frac{k}{n}\right) P_{n,k}(y) \sum_{i=1}^{n-1} u_{n,i}(F) D^2 P_{n,i}(y) dy \right| \\
&= w(x) \left| \sum_{k=1}^{n-1} P_{n,k}(x)\varphi\left(\frac{k}{n}\right) \sum_{i=1}^{n-1} u_{n,i}(F) \int_0^1 DP_{n,k}(y) DP_{n,i}(y) dy \right| \\
&= w(x) \left| \sum_{k=1}^{n-1} P_{n,k}(x)\varphi\left(\frac{k}{n}\right) \sum_{i=1}^{n-1} u_{n,i}(F) \int_0^1 P_{n,k}(y) \frac{k - ny}{\varphi(y)} P_{n,i}(y) \frac{i - ny}{\varphi(y)} dy \right| \\
&\leq S_n(\gamma_0, \gamma_1; x) \| wF \|
\end{aligned} \tag{4.4}$$

with

$$S_n(\gamma_0, \gamma_1; x)$$
$$= w(x) \sum_{k=1}^{n-1} P_{n,k}(x)\varphi\left(\frac{k}{n}\right) \sum_{i=1}^{n-1} u_{n,i}\left(\frac{1}{w}\right) \int_0^1 P_{n,k}(y) \frac{|k - ny|}{\varphi(y)} P_{n,i}(y) \frac{|i - ny|}{\varphi(y)} dy.$$

The next three estimates follow from Hölder inequality.

$$S_n(\gamma_0,\gamma_1;x) \le S_n(-1,\gamma_1;x)^{-\gamma_0} S_n(0,\gamma_1;x)^{1+\gamma_0}; \tag{4.5}$$
$$S_n(-1,\gamma_1;x) \le S_n(-1,-1;x)^{-\gamma_1} S_n(-1,0;x)^{1+\gamma_1}; \tag{4.6}$$
$$S_n(0,\gamma_1;x) \le S_n(0,-1;x)^{-\gamma_1} S_n(0,0;x)^{1+\gamma_1}. \tag{4.7}$$

Applying (4.6) and (4.7) in (4.5) we get

$$\begin{aligned} S_n(\gamma_0,\gamma_1;x) \le{} & S_n(-1,-1;x)^{\gamma_0\gamma_1} S_n(-1,0;x)^{-\gamma_0(1+\gamma_1)} \\ & \times S_n(0,-1;x)^{-\gamma_1(1+\gamma_0)} S_n(0,0;x)^{(1+\gamma_0)(1+\gamma_1)}. \end{aligned} \tag{4.8}$$

Inequalities (4.4) and (4.8) imply that it is enough to prove

$$S_n(\gamma_0,\gamma_1;x) \le \sqrt{2} \tag{4.9}$$

in the four extreme cases $(\gamma_0,\gamma_1) = (0,0),(-1,0),(0,-1),(-1,-1)$ in order to establish the lemma. Applying Cauchy's inequality we get

$$S_n(\gamma_0,\gamma_1;x) \le w(x) \sum_{k=1}^{n-1} P_{n,k}(x)\varphi\left(\frac{k}{n}\right) \sqrt{E_{n,k}(w)}\sqrt{F_{n,k}} \tag{4.10}$$

with

$$E_{n,k}(w) = \sum_{i=1}^{n-1} \int_0^1 \varphi(y)^{-2} P_{n,k}(y) P_{n,i}(y) u_{n,i}^2(w^{-1})\, dy$$
$$F_{n,k} = \sum_{i=1}^{n-1} \int_0^1 \varphi(y)^{-2} P_{n,k}(y)(k-ny)^2 P_{n,i}(y)(i-ny)^2\, dy.$$

For the estimate of $F_{n,k}$ we use properties of the Bernstein operators and (4.3) and get

$$\begin{aligned} F_{n,k} &= \int_0^1 \varphi(y)^{-2} P_{n,k}(y)(k-ny)^2 \left(\sum_{i=1}^{n-1} P_{n,i}(y)(i-ny)^2 \right) dy \\ &\le \int_0^1 \varphi(y)^{-2} P_{n,k}(y)(k-ny)^2 n\varphi(y) dy \\ &= n\int_0^1 (k-ny)d\, P_{n,k}(y) = n^2 \int_0^1 P_{n,k}(y)\, dy = \frac{n^2}{n+1}. \end{aligned} \tag{4.11}$$

Now we estimate $E_{n,k}(w)$ separately in the four extreme cases.

(I) Let $\gamma_0 = \gamma_1 = -1$. In this case $w(x) = \varphi(x)^{-1}$. From (4.3) with $k = i$ we get

$$u_{n,i}(w^{-1}) = u_{n,i}(\varphi) = \int_0^1 n\varphi\left(\frac{i}{n}\right) P_{n,i}(y)\, dy = \frac{n}{n+1}\varphi\left(\frac{i}{n}\right).$$

Using the above equality and (4.3) with $k = i$ we get

$$
\begin{aligned}
E_{n,k}(\varphi^{-1}) &= \int_0^1 \frac{P_{n,k}(y)}{\varphi^2(y)} \sum_{i=1}^{n-1} P_{n,i}(y) u_{n,i}^2(\varphi) dy \\
&= \left(\frac{n}{n+1}\right)^2 \int_0^1 \frac{P_{n,k}(y)}{\varphi^2(y)} \sum_{i=1}^{n-1} P_{n,i}(y) \varphi^2\left(\frac{i}{n}\right) dy \\
&= \frac{n(n-1)}{(n+1)^2} \int_0^1 \frac{P_{n,k}(y)}{\varphi(y)} \sum_{i=1}^{n-1} P_{n-2,i-1}(y) \varphi\left(\frac{i}{n}\right) dy \\
&= \frac{n(n-1)}{(n+1)^2} \int_0^1 \frac{P_{n,k}(y)}{\varphi(y)} \sum_{j=0}^{n-2} P_{n-2,j}(y) \varphi\left(\frac{j+1}{n}\right) dy. \qquad (4.12)
\end{aligned}
$$

Using

$$
\sum_{j=0}^{n-2} P_{n-2,j}(y) \varphi\left(\frac{j+1}{n}\right) = \frac{(n-2)(n-3)}{n^2}\varphi(y) + \frac{n-1}{n}
$$

in (4.12) we get

$$
\begin{aligned}
E_{n,k}(\varphi^{-1}) &= \frac{(n-1)}{n(n+1)^2} \int_0^1 \frac{P_{n,k}(y)}{\varphi(y)} \left((n-2)(n-3)\varphi(y) + n - 1\right) dy \\
&= \frac{(n-1)}{n(n+1)^2} \left(\frac{(n-2)(n-3)}{n+1} + (n-1)\frac{n}{k(n-k)}\right) \\
&\le \frac{(n-1)}{n(n+1)^2} \left(\frac{(n-2)(n-3)}{n+1} + n\right) < \frac{2}{n}.
\end{aligned}
$$

Applying (4.11) and the above estimate in (4.10) we get

$$
S_n(-1,-1;x) \le \sqrt{2}\varphi(x)^{-1} \sum_{k=1}^{n-1} P_{n,k}(x)\varphi(\frac{k}{n}) < \sqrt{2},
$$

which proves (4.9) in the case $\gamma_0 = \gamma_1 = -1$.

(II) Let $\gamma_0 = -1, \gamma_1 = 0$. In this case $w(x) = x^{-1}$ and $u_{n,i}(w^{-1}) = u_{n,i}(x) = i/n$. We shall prove

$$
E_{n,k}(w) \le \frac{2(n+1)}{(n-k)^2}. \qquad (4.13)
$$

For $1 \le k \le n-2$ (hence $n \ge 3$) we have

$$
\begin{aligned}
E_{n,k}(w) &= \int_0^1 \frac{P_{n,k}(y)}{\varphi^2(y)} \sum_{i=1}^{n-1} P_{n,i}(y) \left(\frac{i}{n}\right)^2 dy \\
&\le \int_0^1 \frac{P_{n,k}(y)}{\varphi^2(y)} \left(y^2 + \frac{\varphi(y)}{n}\right) dy = \frac{(n-1)(k+1)}{k(n-k)(n-k-1)} \le \frac{2(n+1)}{(n-k)^2}.
\end{aligned}
$$

For $k = n-1$ (hence $n \geq 2$) we have

$$E_{n,n-1}(w) = \int_0^1 \frac{P_{n,n-1}(y)}{\varphi^2(y)} \sum_{i=1}^{n-1} P_{n,i}(y) \left(\frac{i}{n}\right)^2 dy$$

$$= \int_0^1 \frac{P_{n,n-1}(y)}{\varphi^2(y)} \left(y^2 + \frac{\varphi(y)}{n} - y^n\right) dy = n \int_0^1 \left(\frac{y^{n-2}}{n} + y^{n-1} \sum_{i=0}^{n-3} y^i\right) dy$$

$$\leq n \int_0^1 \left(\frac{y^{n-2}}{n} + y^{n-1}(n-2)\right) dy = \frac{1}{n-1} + n - 2 \leq 2(n+1).$$

This establishes (4.13). Applying (4.11) and (4.13) in (4.10) we get

$$S_n(-1,0;x) \leq x^{-1} \sum_{k=1}^{n-1} P_{n,k}(x) \varphi\left(\frac{k}{n}\right) \sqrt{\frac{2(n+1)}{(n-k)^2}} \sqrt{\frac{n^2}{n+1}}$$

$$= \sqrt{2} x^{-1} \sum_{k=1}^{n-1} P_{n,k}(x) \frac{k}{n} < \sqrt{2},$$

which proves (4.9) in the case $\gamma_0 = -1, \gamma_1 = 0$.

(**III**) The case $\gamma_0 = 0, \gamma_1 = -1$ is symmetric to (**II**) and (4.9) is established in the same way.

(**IV**) Let $\gamma_0 = \gamma_1 = 0$. In this case $w(x) = 1$ and $u_{n,i}(w^{-1}) = 1$. For $1 \leq k \leq n-1$ we have

$$E_{n,k}(1) = \int_0^1 \frac{P_{n,k}(y)}{\varphi^2(y)} \sum_{i=1}^{n-1} P_{n,i}(y) dy = \int_0^1 \frac{P_{n,k}(y)}{\varphi(y)} \left(1 - y^n - (1-y)^n\right) dy$$

$$= \binom{n}{k} \int_0^1 y^{k-1}(1-y)^{n-k-1} \sum_{i=0}^{n-2} \left(y^i + (1-y)^i\right) dy$$

$$= \binom{n}{k} \sum_{i=0}^{n-2} \int_0^1 \left(y^{k-1+i}(1-y)^{n-k-1} + y^{k-1}(1-y)^{n-k-1+i}\right) dy$$

$$= \frac{n}{k(n-k)} \left(2 + \sum_{i=1}^{n-2} \left(\prod_{s=0}^{i-1} \frac{k+s}{n+s} + \prod_{s=0}^{i-1} \frac{n-k+s}{n+s}\right)\right)$$

$$\leq \frac{n}{k(n-k)} \sum_{i=0}^{n-2} \left(\left(\frac{k+n-2}{2n-2}\right)^i + \left(\frac{2n-k-2}{2n-2}\right)^i\right)$$

$$\leq \frac{n}{k(n-k)} \left(\frac{2n-2}{n-k} + \frac{2n-2}{k}\right) = \frac{2n^2(n-1)}{k^2(n-k)^2}.$$

Applying (4.11) and the above estimate in (4.10) we get

$$S_n(0,0;x) \le \sum_{k=1}^{n-1} P_{n,k}(x)\varphi\left(\frac{k}{n}\right)\sqrt{\frac{2n^2(n-1)}{k^2(n-k)^2}}\sqrt{\frac{n^2}{n+1}}$$

$$= \sqrt{2\frac{n-1}{n+1}}\sum_{k=1}^{n-1} P_{n,k}(x) < \sqrt{2},$$

which proves (4.9) in the case $\gamma_0 = \gamma_1 = 0$ and completes the proof of the lemma. □

Proof of Theorem 1.3. We follow the scheme for proving strong inverse theorems of type A given in [2]. Applying Lemma 4.1 with $g = U_n^4 f$, Lemma 2.2 with $g = U_n^3 f$ and $g = U_n^2 f$ and Lemma 4.2 with $F = \varphi D^2(U_n^2 f)$ we get

$$\left\| w\left(U_n^5 f - U_n^4 f - \frac{1}{n}\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right)\right\| \le \frac{\kappa(n)}{4n^2}\left\|w\varphi D^2\left(\varphi D^2(U_n^4 f)\right)\right\|$$

$$= \frac{\kappa(n)}{4n^2}\left\|w\varphi D^2\left(U_n^2\left(\varphi D^2(U_n^2 f)\right)\right)\right\| \le \frac{\sqrt{2}\kappa(n)}{4n}\|w\varphi D^2(U_n^2 f)\|.$$

Using the last inequality, Lemma 4.2 with $F = f - U_n^3 f$ and with $F = f - U_n^2 f$ and Lemma 2.3 we get

$$\left\| w\left(U_n^5 f - U_n^4 f - \frac{1}{n}\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right)\right\|$$

$$\le \frac{\sqrt{2}\kappa(n)}{4n}\left\|w\varphi D^2 U_n^2\left(f - \frac{U_n^2 f + U_n^3 f}{2}\right)\right\| + \frac{\sqrt{2}\kappa(n)}{4n}\left\|w\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right\|$$

$$\le \frac{\sqrt{2}\kappa(n)}{8n}\left\|w\varphi D^2\left(U_n^2\left(f - U_n^2 f\right)\right)\right\| + \frac{\sqrt{2}\kappa(n)}{8n}\left\|w\varphi D^2\left(U_n^2\left(f - U_n^2 f\right)\right)\right\|$$

$$+ \frac{\sqrt{2}\kappa(n)}{4n}\left\|w\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right\|$$

$$\le \frac{\kappa(n)}{4}\left(\|w(U_n^2 f - f)\| + \|w(U_n^3 f - f)\|\right) + \frac{\sqrt{2}\kappa(n)}{4n}\left\|w\varphi D^2\left(\frac{U_n^5 f + U_n^4 f}{2}\right)\right\|$$

$$\le \frac{5\kappa(n)}{4}\|w(U_n f - f)\| + \frac{\sqrt{2}\kappa(n)}{4n}\left\|w\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right\|.$$

From the above inequality and Lemma 2.3 we have

$$\frac{1}{n}\left\|w\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right\|$$

$$\le \left\| w\left(U_n^5 f - U_n^4 f - \frac{1}{n}\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right)\right\| + \|w(U_n^5 f - U_n^4 f)\|$$

$$\le \frac{4+5\kappa(n)}{4}\|w(U_n f - f)\| + \frac{\sqrt{2}\kappa(n)}{4n}\left\|w\varphi D^2\left(\frac{U_n^4 f + U_n^5 f}{2}\right)\right\|,$$

which can be rewritten as

$$\frac{1}{2n}\left\|w\varphi D^2\left(\frac{U_n^4f+U_n^5f}{2}\right)\right\| \le \frac{4+5\kappa(n)}{8-2\sqrt{2}\kappa(n)}\|w(U_nf-f)\|. \tag{4.14}$$

Finally from the definition of K-functional, Lemma 2.3 and (4.14) we obtain

$$\begin{aligned}K_w\left(f,\frac{1}{2n}\right) &= \inf_{g\in W^2(w\varphi)}\left\{\|w(f-g)\|+\frac{1}{2n}\|w\varphi D^2g\|\right\}\\ &\le \left\|w\left(f-\left(\frac{U_n^4f+U_n^5f}{2}\right)\right)\right\|+\frac{1}{2n}\left\|w\varphi D^2\left(\frac{U_n^5f+U_n^4f}{2}\right)\right\|\\ &\le \left(\frac{9}{2}+\frac{4+5\kappa(n)}{8-2\sqrt{2}\kappa(n)}\right)\|w(f-U_nf)\|\\ &= \frac{40-(9\sqrt{2}-5)\kappa(n)}{8-2\sqrt{2}\kappa(n)}\|w(f-U_nf)\|.\end{aligned}$$

Theorem 1.3 is proved. □

Remark 4.1. It is essential that we consider the derivatives of U_n^2F in the Bernstein-type inequality in Lemma 4.2. The analogous inequality for U_nF is

$$n^{-1}\|w\varphi D^2(U_nF)\| \le 4\|wF\|$$

and the constant 4 cannot be improved if either $\gamma_0=1$ or $\gamma_1=1$. This constant is too big and the technique used in the proof of Theorem 1.3 does not work with $D^2(U_nF)$.

References

[1] H. Berens, Y. Xu, On Bernstein-Durrmeyer Polinonials with Jacobi-weights. *in "Approximation Theory and Functional Analysis", ed. C.K. Chui, Academic Press, Boston,* (1991) 25-46.

[2] Z. Ditzian and K. G. Ivanov, Strong Convers Inequalities. *J. d'Analyse Mathematique,61,* (1991), 61-111.

[3] B. R. Draganov and K. G. Ivanov. A new characterization of weighted Peetre K-functionals (III). *manuscript.*

[4] T. N. T. Goodman and A. Sharma, A Bernstein-type Operator on the Simplex. *Mathematica Balcanica (new series) 5,2,* (1991), 129-145.

[5] T. N. T. Goodman and A. Sharma, A Modified Bernstein-Shoenberg Operator. *Proc. of Conference on Constructive Theory of Functions, Varna'87, Publ. House of Bulg. Acad. of Sci., Sofia,* (1987), 166-173.

[6] P. E. Parvanov and B. D. Popov, The Limit Case of Bernstein's Operators with Jacobi-weights. *Mathematica Balcanica (new series), Vol.8, Fask.2-3,* (1994), 165-177.