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Weighted Approximation by Meyer-König and Zeller-Type Operators

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The uniform weighted approximation errors of a Goodman-Sharma variant of the Meyer-König and Zeller operators are characterized for functions from $C(w)[0,1)+W^2(w\varphi)[0,1)$ with $\varphi(x)=x(1-x)^2$ and weight of the form $w(x)=x^{\alpha_0}(1-x)^{\alpha_1}$ for $\alpha_0,\alpha_1\in[-1,0]$. Direct and strong converse theorems of type A are proved in terms of the weighted K-functional.

Keywords and Phrases: Meyer-König and Zeller operators, Baskakov operators, direct theorem, strong converse theorem, K-functional.

1. Introduction

The Meyer-König and Zeller operators (introduced in 1960 [18]) in the slight modification of Cheney and Sharma [3] are defined for $f\in C[0,1)$ by

$$M_n^{[MKZ]}f(x)=\sum_{k=0}^{\infty}P_{n,k}(x)f\Big(\frac{k}{k+n}\Big),\qquad x\in[0,1),\tag{1}$$

where $P_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}$. In the paper we shall refer to the operators $M_n^{[MKZ]}$ in (1) as MKZ operators. They were extensively studied in approximation theory, as many direct and converse results were proved (see e.g. [20]).

Following the Durrmeyer modification [7, 5] of Bernstein polynomials, many authors modified MKZ operators for Lebesgue integrable functions on $(0,1)$.

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Heilmann [14, 15] and Abel, Gupta and Ivan [1] unified these Durrmeyer-type modifications in the following family of operators

$$\tilde{M}_{n,\nu,\kappa}f(x) = \sum_{k=\max\{0,\kappa\}}^{\infty} P_{n,k}(x)(n+\nu)\int_0^1 P_{n+\nu,k-\kappa}(y)f(y)\frac{dy}{(1-y)^2} \tag{2}$$

defined for all positive integers n satisfying $n+\nu > 0$, $\nu \in \mathbb{R}$ and $\kappa \in \mathbb{Z}$. Special cases of (2) are the operators: $\tilde{M}_{n,2,0}$ introduced by Chen [4]; $\tilde{M}_{n-1,0,2}$ introduced by Guo [11]; $\tilde{M}_{n,\nu,0}$ introduced by Heilmann [13]; $\tilde{M}_{n-1,1,0}$ introduced by Gupta and Abel [12]. In [14] Heilmann called $\tilde{M}_{n,0,0}$ the "most natural" MKZ-Durrmeyer operator.

The Goodman-Sharma-type modification of MKZ operators (GS-MKZ) discussed in this paper is given for natural number n by

$$\begin{gathered} M_n f(x) = \sum_{k=0}^{\infty} P_{n,k}(x)u_{n,k}(f), \\ u_{n,0}(f) = f(0), \qquad u_{n,k}(f) = n\int_0^1 P_{n,k-1}(y)f(y)\frac{dy}{(1-y)^2}, \end{gathered} \tag{3}$$

where f is a Lebesgue integrable in $(0,1)$ function with a finite limit $f(0)$ at 0. In the definition of GS-MKZ operators we follow the idea of Goodman and Sharma [9, 10] (see also [19]) for a modification of Bernstein-Durrmeyer operators which preserves linear functions. In the same way the operators defined in [8] by Finta (see also [17]) are a Goodman-Sharma-type modification of the Baskakov-Durrmeyer operators. In the terminology of (2) we have

$$M_n f(x) = P_{n,0}(x)f(0) + \tilde{M}_{n,0,1}f(x).$$

GS-MKZ operators combine good properties of both MKZ operators and their Durrmeyer modifications. Thus, GS-MKZ operators like $M_n^{[MKZ]}$ operators preserve linear functions and are suitable for uniform approximation. Moreover, M_n like MKZ-Durrmeyer operators commute among themselves, i.e. $M_n M_k f = M_k M_n f$, and with the differential operator φD^2.

The first derivative operator is given by $D = \frac{d}{dx}$, thus $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$. By $\varphi(x) = x(1-x)^2$ we denote the weight which is naturally connected with the second derivatives of both MKZ and GS-MKZ operators. By weight we mean a non-negative measurable function w. In particular, we consider the weight functions

$$w(x) = w(\alpha_0,\alpha_1;x) = x^{\alpha_0}(1-x)^{\alpha_1}, \qquad x \in (0,1), \tag{4}$$

defined for real values of the parameters α_0, α_1. Our main results will concern the values of the powers α_0, α_1 in the range $[-1,0]$.

By $C[0,1)$ we denote the space of all continuous on $[0,1)$ functions. The functions from $C[0,1)$ are not expected to be continuous or bounded at 1.

By $L_\infty[0,1)$ we denote the space of all Lebesgue measurable and essentially bounded in $[0,1)$ functions equipped with the uniform norm $\|\cdot\|_{[0,1)}$. For a weight function w we set $C(w)[0,1) = \{f \in C[0,1) : wf \in L_\infty[0,1)\}$ and $W^2(w\varphi)[0,1) = \{g, g' \in AC_{loc}(0,1) : w\varphi D^2 g \in L_\infty[0,1)\}$, where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$.

Set $C_0(w)[0,1) = \{f \in C(w)[0,1) : f(0) = 0\}$. Similarly, by $W_0^2(w\varphi)[0,1)$ we denote the subspace of $W^2(w\varphi)[0,1)$ of functions g satisfying the additional boundary conditions $\lim_{x\to 0+0} \varphi(x)D^2 g(x) = 0$. Note that the boundary conditions for both $C_0(w)$ and $W_0^2(w\varphi)$ do not depend on the weight w. These conditions are essential only when the weight w does not go to ∞ at 0, while for $\alpha_0 < 0$ we have $C_0(w)[0,1) = C(w)[0,1)$ and $W_0^2(w\varphi)[0,1) = W^2(w\varphi)[0,1)$.

The weighted approximation error $\|w(f - M_n f)\|$ of M_n will be compared with the K-functional between the weighted spaces $C(w)$ and $W^2(w\varphi)$, which for every

$$f \in C(w)[0,1) + W^2(w\varphi)[0,1) = \{f_1 + f_2 : f_1 \in C(w)[0,1),\ f_2 \in W^2(w\varphi)[0,1)\}$$

and $t > 0$ is defined by

$$K_w(f,t)_{[0,1)} = \inf_{g\in W^2(w\varphi),\ f-g\in C(w)} \{\|w(f-g)\|_{[0,1)} + t\|w\varphi D^2 g\|_{[0,1)}\}. \quad (5)$$

Our main result is the following theorem, consisting of a direct inequality and a strong converse inequality of type A in the terminology of [6].

Theorem 1. *Let $w = w(\alpha_0, \alpha_1)$ be given by (4) with $\alpha_0, \alpha_1 \in [-1,0]$. Then for every $f \in C(w)[0,1) + W^2(w\varphi)[0,1)$ and every $n \in \mathbb{N}$, $n \geq 4$, we have*

$$\|w(f - M_n f)\|_{[0,1)} \leq 2\, K_w\left(f, 1/(2n)\right)_{[0,1)} \leq 13.7\, \|w(f - M_n f)\|_{[0,1)}.$$

Theorem 1 is proved in Section 2 by transferring the corresponding result for the Goodman–Sharma-type modification V_n of Baskakov operators given in [17]. The transformation $\mathscr{T}$ defined in (7) relates the results for the two types of operators on the base of Proposition 8 where the similarity of M_n and V_n is established. With the help of Proposition 8 we also obtain many other properties of M_n in Section 3. We emphasize that the proper connection between the operators of MKZ-type and of Baskakov-type requires the utilization of *truly weighted* approximations, as explained in Remark 2. The similarity of the MKZ operators $M_n^{[MKZ]}$ and the original Baskakov operators is also provided by transformation $\mathscr{T}$ as shown in Proposition 7.

Finally, we make some observations on the spaces $C(w)[0,1)+W^2(w\varphi)[0,1)$ when the weight w satisfies the assumptions of Theorem 1. If $-1 < \alpha_1 < 0$ and $\alpha_0 \in [-1,0]$ then $C(w)[0,1) + W^2(w\varphi)[0,1) = C(w)[0,1) + \pi_1$, where π_1 is the set of all algebraical polynomials of degree 1. Note that π_1 is the null space of the operator φD^2.

But for $\alpha_1 = 0$ or for $\alpha_1 = -1$ the space $C(w)[0,1) + W^2(w\varphi)[0,1)$ is essentially bigger than $C(w)[0,1) + \pi_1$ for every $\alpha_0 \in [-1,0]$. Thus, if

$$f(x) = \begin{cases} e^{-1}x, & x \in [0, e/(1+e)], \\ (1-x)(\ln x - \ln(1-x)), & x \in [e/(1+e), 1], \end{cases}$$

then $f \in W^2(w\varphi)[0,1) \backslash (C(w)[0,1) + \pi_1)$ for $\alpha_1 = -1$ and $\alpha_0 \in [-1,0]$. Also, if

$$f(x) = \begin{cases} e^{-1}x^2(1-x)^{-1}, & x \in [0, e/(1+e)], \\ x(\ln x - \ln(1-x)), & x \in [e/(1+e), 1], \end{cases}$$

then $f \in W^2(w\varphi)[0,1) \backslash (C(w)[0,1) + \pi_1)$ for $\alpha_1 = 0$ and $\alpha_0 \in [-1,0]$.

The non-emptiness of $(C(w)[0,1) + W^2(w\varphi)[0,1)) \backslash (C(w)[0,1) + \pi_1)$ is determined by the decrease of $\varphi(x)$ as $(1-x)^2$ at $1-0$. In this respect GS-MKZ operators behave differently from Goodman–Sharma operators (see [16]), where for all weights w we have $C(w)[0,1] + W^2(w\psi)[0,1] = C(w)[0,1] + \pi_1$ (with the appropriate weight $\psi(x) = x(1-x)$ for these operators).

2. A Connection Between the Goodman–Sharma Modifications of Baskakov and MKZ Operators

In this section we study a simple transformation $\mathscr{T}$ mapping functions defined on $[0,\infty)$ into functions defined on $[0,1)$. Operator $\mathscr{T}$ will allow us to relate results for Goodman–Sharma modification of Baskakov operators to their counterparts for Goodman–Sharma modification of MKZ operators and vice versa.

We shall consider variables and functions defined in $[0,1)$ and their analogs defined in $[0,\infty)$, as the later will be denoted with tilde. The first derivative operator in $[0,1)$ is given by $D = d/dx$, while the same operator (with respect to $\tilde{x}$) in $[0,\infty)$ is denoted by $\tilde{D} = d/d\tilde{x}$. We use $\|\cdot\|_{[0,\infty)}$ for the supremum norm in $[0,\infty)$ and the sets $C(\tilde{w})[0,\infty)$, $C_0(\tilde{w})[0,\infty)$, $W^2(\tilde{w}\tilde{\varphi})[0,\infty)$, $W_0^2(\tilde{w}\tilde{\varphi})[0,\infty)$ have analogous definitions to the spaces $C(w)[0,1)$, $C_0(w)[0,1)$, $W^2(w\varphi)[0,1)$, $W_0^2(w\varphi)[0,1)$.

We shall utilize the change of variable $\sigma : [0,1) \to [0,\infty)$ given by

$$\tilde{x} = \sigma(x) = \frac{x}{1-x}. \tag{6}$$

Then its inverse change of variable $\sigma^{-1} : [0,\infty) \to [0,1)$ is

$$x = \sigma^{-1}(\tilde{x}) = \frac{\tilde{x}}{1+\tilde{x}}.$$

A function $\tilde{f}$ defined on $[0,\infty)$ is transformed to a function f defined on $[0,1)$ by

$$f(x) = \mathscr{T}(\tilde{f})(x) = \lambda(x)(\tilde{f} \circ \sigma)(x), \qquad \lambda(x) = 1-x. \tag{7}$$

Then the inverse operator $\mathscr{T}^{-1}$ transforming a function f defined in $[0,1)$ to a function $\tilde{f}$ defined on $[0,\infty)$ is

$$\tilde{f}(\tilde{x}) = \mathscr{T}^{-1}(f)(\tilde{x}) = \frac{1}{(\lambda\circ\sigma^{-1})(\tilde{x})}(f\circ\sigma^{-1})(\tilde{x}).$$

When a product of two functions is treated we shall utilize for the "weight" functions the associated operator

$$w(x) = \mathscr{S}(\tilde{w})(x) = \frac{1}{\lambda(x)}(\tilde{w}\circ\sigma)(x) \tag{8}$$

(in order to compensate for the multiplier λ in the definition of $\mathscr{T}$) and its inverse

$$\tilde{w}(\tilde{x}) = \mathscr{S}^{-1}(w)(\tilde{x}) = (\lambda\circ\sigma^{-1})(\tilde{x})(w\circ\sigma^{-1})(\tilde{x}).$$

With these notations for $f = \mathscr{T}(\tilde{f})$ and $w = \mathscr{S}(\tilde{w})$ we have

$$\begin{aligned} wf &= \mathscr{S}(\tilde{w})\mathscr{T}(\tilde{f}) = (\tilde{w}\circ\sigma)(\tilde{f}\circ\sigma),\\ \tilde{w}\tilde{f} &= \mathscr{S}^{-1}(w)\mathscr{T}^{-1}(f) = (w\circ\sigma^{-1})(f\circ\sigma^{-1}). \end{aligned} \tag{9}$$

From (9) and (6) we get

$$\int_0^\infty \tilde{w}(\tilde{y})\tilde{f}(\tilde{y})\,d\tilde{y} = \int_0^1 \mathscr{S}(\tilde{w})(y)\mathscr{T}(\tilde{f})(y)\frac{dy}{(1-y)^2}. \tag{10}$$

The differential operator naturally connected with both MKZ and GS-MKZ operators is φD^2 with weight $\varphi(x) = x(1-x)^2$. For Baskakov operators the similar differential operator is $\tilde{\varphi}\tilde{D}^2$ with $\tilde{\varphi}(\tilde{x}) = \tilde{x}(1+\tilde{x})$. Important connections between these two differential operators and operators $\mathscr{T}$ and $\mathscr{T}^{-1}$ are

$$\mathscr{T}(\tilde{\varphi}\tilde{D}^2\tilde{g}) = \varphi D^2(\mathscr{T}\tilde{g}), \qquad \mathscr{T}^{-1}(\varphi D^2 g) = \tilde{\varphi}\tilde{D}^2(\mathscr{T}^{-1}g). \tag{11}$$

Now, we list several properties of operators $\mathscr{T}$ and $\mathscr{S}$. Definition (7) immediately yields

Proposition 1. *Let $\mathcal{F}[0,1)$ and $\tilde{\mathcal{F}}[0,\infty)$ denote the spaces of all functions defined in $[0,1)$ and $[0,\infty)$, respectively. Then $\mathscr{T}:\tilde{\mathcal{F}}[0,\infty)\to\mathcal{F}[0,1)$ and $\mathscr{T}^{-1}:\mathcal{F}[0,1)\to\tilde{\mathcal{F}}[0,\infty)$ are linear positive operators.*

Set $e_k(x) = x^k$ and $\tilde{e}_k(\tilde{x}) = \tilde{x}^k$ for $k = 0,1$. From $\mathscr{T}(\tilde{e}_0) = e_0 - e_1$, $\mathscr{T}(\tilde{e}_1) = e_1$ and Proposition 1 we obtain

Proposition 2. *Let π_1 and $\tilde{\pi}_1$ denote the sets of linear functions in $[0,1)$ and $[0,\infty)$, respectively. Then $\mathscr{T}:\tilde{\pi}_1\to\pi_1$ is an one-to-one correspondence.*

Proposition 3. *(i) f is concave in $[0,1)$ iff $\mathscr{T}^{-1}(f)$ is concave in $[0,\infty)$;*
(ii) f is convex in $[0,1)$ iff $\mathscr{T}^{-1}(f)$ is convex in $[0,\infty)$;
(iii) $1/w$ is concave in $[0,1)$ iff $1/\mathscr{S}^{-1}(w)$ is concave in $[0,\infty)$;
(iv) $1/w$ is convex in $[0,1)$ iff $1/\mathscr{S}^{-1}(w)$ is convex in $[0,\infty)$.

Proof. Statements (i) and (ii) follow from the definitions of concavity and convexity and Propositions 1 and 2. Statements (iii) and (iv) follow from $1/\mathscr{S}^{-1}(w) = \mathscr{T}^{-1}(1/w)$ and statements (i) and (ii). □

From (7) and (9) we get

Proposition 4. *Let w be a weight in $[0,1)$ and $\tilde{w} = \mathscr{S}^{-1}(w)$. Then the mapping $\mathscr{T} : C(\tilde{w})[0,\infty) \to C(w)[0,1)$ is an one-to-one correspondence with*

$$\|w\mathscr{T}(\tilde{f})\|_{[0,1)} = \|\tilde{w}\tilde{f}\|_{[0,\infty)}, \qquad \|\tilde{w}\mathscr{T}^{-1}(f)\|_{[0,\infty)} = \|wf\|_{[0,1)}.$$

From (7), (9) and (11) it follows

Proposition 5. *Let w be a weight in $[0,1)$ and $\tilde{w} = \mathscr{S}^{-1}(w)$. Then the mapping $\mathscr{T} : W^2(\tilde{w}\tilde{\varphi})[0,\infty) \to W^2(w\varphi)[0,1)$ is an one-to-one correspondence with*

$$\|w\varphi D^2(\mathscr{T}(\tilde{f}))\|_{[0,1)} = \|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{f}\|_{[0,\infty)}, \ \ \|\tilde{w}\tilde{\varphi}\tilde{D}^2(\mathscr{T}^{-1}(f))\|_{[0,\infty)} = \|w\varphi D^2 f\|_{[0,1)}.$$

Now we define the K-functional for $\tilde{f} \in C(\tilde{w})[0,\infty) + W^2(\tilde{w}\tilde{\varphi})[0,\infty)$ and $t > 0$ by analogy with (5):

$$K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)} = \inf_{\tilde{g}\in W^2(\tilde{w}\tilde{\varphi}),\ \tilde{f}-\tilde{g}\in C(\tilde{w})} \left\{\|\tilde{w}(\tilde{f}-\tilde{g})\|_{[0,\infty)} + t\|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g}\|_{[0,\infty)}\right\}.$$

Applying Proposition 1, Proposition 4, and Proposition 5 we obtain

Proposition 6. *Let w be a weight in $[0,1)$ and $\tilde{w} = \mathscr{S}^{-1}(w)$. Then for every $f \in C(w)[0,1) + W^2(w\varphi)[0,1)$, $\tilde{f} = \mathscr{T}^{-1}f$ and $t > 0$ we have*

$$K_w(f,t)_{[0,1)} = K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)}.$$

The following proposition gives the connection between the MKZ operators $M_n^{[MKZ]}$ and the classical Baskakov operators [2]

$$B_n\tilde{f}(\tilde{x}) = \sum_{k=0}^{\infty} \tilde{P}_{n,k}(\tilde{x})\tilde{f}\Big(\frac{k}{n}\Big), \quad \tilde{P}_{n,k}(\tilde{x}) = \binom{n+k-1}{k}\tilde{x}^k(1+\tilde{x})^{-n-k}. \quad (12)$$

Proposition 7. *For every f such that one of the series in (13) is convergent and for every $n \in \mathbb{N}$ we have*

$$M_n^{[MKZ]}(f)(x) = \mathscr{T}(B_n(\mathscr{T}^{-1}(f)))(x), \qquad x \in [0,1). \quad (13)$$

Proof. From (12), Proposition 1, (1) and the identities

$$\frac{n+k}{n}\mathscr{T}(\tilde{P}_{n,k})(x) = P_{n,k}(x), \qquad \mathscr{T}^{-1}(f)\Big(\frac{k}{n}\Big) = \frac{n+k}{n}f\Big(\frac{k}{n+k}\Big) \quad (14)$$

valid for $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ we obtain (13). □

Finta [8] introduced in 2005 the operators

$$\begin{gathered} V_n\tilde{f}(\tilde{x}) = \sum_{k=0}^{\infty} \tilde{P}_{n,k}(\tilde{x}) v_{n,k}(\tilde{f}), \\ v_{n,0}(\tilde{f}) = \tilde{f}(0), \qquad v_{n,k}(\tilde{f}) = (n+1)\int_0^{\infty} \tilde{P}_{n+2,k-1}(\tilde{y})\tilde{f}(\tilde{y})\, d\tilde{y}, \end{gathered} \tag{15}$$

where $\tilde{P}_{n,k}$ are given in (12). $V_n\tilde{f}$ modify Baskakov operators similarly to the Goodman–Sharma modification of Bernstein operators. Operators (3) and (15) are related by

Proposition 8. *For every integrable function f such that one of the series in* (16) *is convergent and for every $n \in \mathbb{N}$ we have*

$$M_n(f)(x) = \mathscr{T}(V_n(\mathscr{T}^{-1}(f)))(x), \qquad x \in [0,1). \tag{16}$$

Proof. From (15), (10) with $\tilde{w} = \tilde{P}_{n+2,k-1}$, the identity $(n+1)\mathscr{S}(\tilde{P}_{n+2,k-1}) = (n+k)P_{n,k-1}$, and (3) we get

$$v_{n,k}(\mathscr{T}^{-1}(f)) = \frac{n+k}{n}\, u_{n,k}(f), \qquad k \in \mathbb{N}_0. \tag{17}$$

From (15), Proposition 1, (14), (17) and (3) we get (16). □

Now, we are ready to establish the validity of Theorem 1.

Proof of Theorem 1. From Proposition 8, (9), (7) and (8) we obtain for $\tilde{f} = \mathscr{T}^{-1}f$ and $\tilde{w} = \mathscr{S}^{-1}w$

$$w(f - M_n(f)) = (\tilde{w}(\tilde{f} - V_n(\tilde{f}))) \circ \sigma$$

and hence

$$\|w(f - M_n(f))\|_{[0,1)} = \|\tilde{w}(\tilde{f} - V_n(\tilde{f}))\|_{[0,\infty)}. \tag{18}$$

From Theorem 1.1 in [17], (18) and Proposition 6 we get the assertion in Theorem 1. □

Remark 1. One may consider representations (13) and (16) (with yet unknown operator $\mathscr{T}$) as definitions of the operators $M_n^{[MKZ]}$ and M_n. Then, for preserving of the main properties of B_n or V_n by $M_n^{[MKZ]}$ or M_n, respectively, it is necessary $\mathscr{T}$ to satisfy identity (11). If $\mathscr{T}$ is of type (7), then (11) is fulfilled if and only if λ and σ satisfy

$$D^2\lambda = 0, \qquad 2(D\lambda)(D\sigma) + (D^2\sigma)\lambda = 0 \tag{19}$$

and $\tilde{\varphi}$ is defined as

$$\tilde{\varphi} = ((D\sigma)^2\varphi) \circ \sigma^{-1}. \tag{20}$$

Remark 2. The change of variable σ was systematically used by Totik [20] (and many other afterward, see e.g. [13, 14, 15]) in the study of operators of MKZ type. Instead of $\tilde{f} = \mathscr{T} f$ Totik used $\tilde{f} = f \circ \sigma$ to connect functions defined on $[0, 1)$ and on $[0, \infty)$. It seems the simple operator $\mathscr{T}$ is used *for the first time here* in establishing connections between operators of MKZ type and of Baskakov type. One explanation is given by the fact that the proper connection between the two types of operators requires the utilization of *weighted* approximations. Here are our arguments.

$\mathscr{T}$ is a natural operator to be considered as Propositions 7 and 8 demonstrate. At the same time (19) shows that the unique (up to a constant factor) choice of λ is $\lambda(x) = 1-x$ for the particular form of σ in (6). Then, according to (18), the connection between the weights is necessarily given by the associated operator $\tilde{w} = \mathscr{S}^{-1} w$. In particular, for the weight function $w(x) = w(\alpha_0, \alpha_1; x) = x^{\alpha_0}(1-x)^{\alpha_1}$ from (4) we have

$$\tilde{w}(\tilde{x}) = \mathscr{S}^{-1}(w)(\tilde{x}) = \left(\frac{\tilde{x}}{1+\tilde{x}}\right)^{\alpha_0} (1+\tilde{x})^{-1-\alpha_1}. \tag{21}$$

Hence, the study of unweighted ($w \equiv 1$, $\alpha_0 = \alpha_1 = 0$) approximation properties of MKZ type operators requires the study of Baskakov type operators approximation properties with weight $\tilde{w}(\tilde{x}) = (1+\tilde{x})^{-1}$. Similarly, the corresponding weight to $\tilde{w} \equiv 1$ is $w(x) = (1-x)^{-1}$. Let us point out that, in view of (21), the exponents α_0, α_1 in Theorem 1 and γ_0, γ_∞ in [17, Theorem 1.1] simultaneously belong to $[-1, 0]$, which is exactly the case of concave $1/w$ and $1/\mathscr{S}^{-1}(w)$ (see also Proposition 3(iii)).

Remark 3. The connection between objects defined on $[0, \infty)$ and on $[0, 1)$ is given either by the operator $f = \mathscr{T} \tilde{f}$ (for functions) or by the associated operator $w = \mathscr{S} \tilde{w}$ (for weights). There are two notable exceptions from this rule. The connection between the MKZ and Baskakov basic functions, $P_{n,k}$ and $\tilde{P}_{n,k}$, is given by (14) and the connection between the weights φ and $\tilde{\varphi}$ in the differential operators is given by (20). These exceptions allow Propositions 7 and 8 and identity (9) to be true and the whole scheme to work.

3. Properties of GS–MKZ Operators

In this section we collect some properties of GS-MKZ operators that follow immediately from the results in Section 2 and the respective properties of Goodman-Sharma type modification of Baskakov operators proved in [17].

We first observe that the GS-MKZ operators are well defined for big n's on functions f that may have power-type divergence at 1. More precisely, if $w(0, \alpha_1) f \in L_\infty[0, 1)$ for some positive α_1, then $M_n f$ is defined for every natural $n > \alpha_1$.

From Propositions 8, 1 and 3 and properties (2.14), (2.15) and (2.17) of operators V_n in [17] we get the following properties of operators M_n:

- M_n is a linear, positive operator;
- $M_n e_0(x) = e_0(x)$, $M_n e_1(x) = e_1(x)$ for $e_k(x) = x^k$;
- $M_n f \le f$ for every concave continuous function f.

From [17, Lemma 2.1], Propositions 8 and 1 and identities (11) we get

Proposition 9. *If $f \in C(w(0,\alpha_1))[0,1)$ and $n \in \mathbb{N}$, $n > \alpha_1$, then*

$$M_n f(x) - M_{n+1} f(x) = \frac{1}{n(n+1)}\,\varphi(x) D^2 M_n f(x), \qquad x \in [0,1).$$

From [17, Theorem 2.5], Propositions 8 and 5 and identities (11) we obtain

Proposition 10. *For every $g \in W_0^2\left(w(0,\alpha_1)\varphi\right)[0,1)$ and $n \in \mathbb{N}$, $n > \alpha_1$, we have*

$$\varphi(x) D^2 M_n g(x) = M_n(\varphi D^2 g)(x), \qquad x \in [0,1),$$

i.e. M_n commutes with the operator φD^2 on $W_0^2\left(w(0,\alpha_1)\varphi\right)[0,1)$.

From [17, Theorem 2.6] and Proposition 8 it follows

Proposition 11. *Let $m, n \in \mathbb{N}$, $m, n > \alpha_1$. Then for every $f \in C(w(0,\alpha_1))$ we have $M_n M_m f = M_m M_n f$, i.e. M_m and M_n commute on $C(w(0,\alpha_1))$.*

From [17, Lemma 2.7], Propositions 8 and 3(iii) and identities (9) we obtain

Proposition 12. *Let $1/w$ be concave. Then for every $f \in C(w)[0,1)$ and $n \in \mathbb{N}$ we have $\|w M_n f\|_{[0,1)} \le \|w f\|_{[0,1)}$, i.e. M_n has norm 1 in $C(w)[0,1)$.*

From [17, Lemma 2.8], Propositions 8 and 3(iii) and identities (9) and (11) we get

Proposition 13. *Let $1/w$ be concave. Then for every $g \in W^2(w\varphi)[0,1)$ and $n \in \mathbb{N}$ we have $\|w\varphi D^2 M_n g\|_{[0,1)} \le \|w\varphi D^2 g\|_{[0,1)}$.*

From [17, Lemma 3.2], Propositions 5 and 3(iii) and identities (18) we get

Proposition 14 (Jackson-type inequality). *Let $1/w$ be concave. Then for every $g \in W^2(w\varphi)[0,1)$ and $n \in \mathbb{N}$ we have*

$$\|w(M_n g - g)\|_{[0,1)} \le \frac{1}{n}\|w\varphi D^2 g\|_{[0,1)}.$$

From [17, Lemma 4.1], Propositions 8, 1 and 3(iii) and identities (11) and (9) we get

Proposition 15 (Strong Voronovskaya-type estimate). *Let* $1/w$ *be concave. Then for every* $g \in W_0^2(w\varphi)[0,1)$ *such that* $\varphi D^2 g \in W^2(w\varphi)[0,1)$ *and for every* $n \in \mathbb{N}$ *we have*

$$\Big\| w\Big(M_n g - g - \frac{1}{n}\varphi D^2\Big(\frac{g+M_n g}{2}\Big)\Big)\Big\|_{[0,1)} \le \frac{1}{4n^2}\|w\varphi D^2(\varphi D^2 g)\|_{[0,1)}.$$

From [17, Lemma 4.2], Proposition 8 and identities (11), (21) and (9) we get

Proposition 16 (Bernstein-type inequality). *Let* $w = w(\alpha_0,\alpha_1)$ *be given by* (4) *with* $\alpha_0,\alpha_1 \in [-1,0]$*. For every* $f \in C_0(w)[0,1)$ *and for every* $n \in \mathbb{N}$, $n \ge 4$*, we have*

$$\frac{1}{n}\big\|w\varphi D^2(M_n^2 f)\big\|_{[0,1)} \le \frac{5}{3}\|wf\|_{[0,1)}.$$

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